Existence and Uniqueness of invariant measures: an approach via sectorial forms

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Abstract

We prove existence and uniqueness for invariant measures of strongly continuous semigroups on $L^2(X;\mu)$, where X is a (possibly infinite–dimensional) space. Our approach is purely analytic based on the theory of sectorial forms. The generators covered are e.g. small perturbations (in the sense of sectorial forms) of operators generating hypercontractive semigroups. An essential ingredient of the proofs is a new result on compact embeddings of weighted Sobolev spaces $H^{1,2}(\rho \cdot dx)$ on $\mathbb{R}^d$ (resp. a Riemannian manifold) into $L^2(\rho\, dx)$. Probabilistic consequences are also briefly discussed.

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1 Introduction

In this paper we study the question of existence and uniqueness of invariant measures for diffusions solving stochastic differential equations of type

$$dX_t = \sigma(X_t)\, dW_t + v(X_t)\, dt\ . \tag{1.1}$$

Here $(X_t)_{t\geq 0}$ takes values in a locally convex topological vector space X, $(W_t)_{t\geq 0}$ is a Brownian motion on X, and the operator valued diffusion map σ and the drift $v : X \to X$ are supposed to be merely measurable. This question has been studied by many authors. We refer to the recent book by G. Da Prato and J. Zabczyk ([13]) for references.

Our approach in this paper is analytic and based on a reformulation of the problem in terms of sectorial forms (cf. [16] and also [19]). Thus we avoid to have to solve (1.1) first and to pin down the exact meaning (strong, mild, weak etc.) of "solution" to (1.1). More precisely, the relevant sectorial forms are of type (closure of)

$$\mathcal{E}_v(f,g) := \mathcal{E}(f,g) + \int_X \Big(v(x), \nabla_H f(x)\Big)\ g(x)\ \mu(dx), \tag{1.2}$$

on $L^2(X;\mu)$ with μ a probability measure on X and

$$\mathcal{E}(f,g) := \int_X \Big(A(x)\ \nabla_H f(x), \nabla_H g(x)\Big)_H\ \mu(dx), \tag{1.3}$$

where f, g are bounded smooth cylinder functions on X and H is a separable real Hilbert space densely and continuously embedded into X. H should be thought of as a "tangent space" to X at every point $x \in X$. ∇_H denotes the corresponding gradient. We refer to Subsection 2.2 for details and the precise assumptions on the map $x \mapsto A(x)$ taking values in the space of symmetric positive definite bounded linear operators on H. We only mention here that $\mathcal{E}_v$ is assumed to be a form perturbation of $\mathcal{E}$ (i.e., v satisfies (2.12) below). In particular, $D(\mathcal{E}) = D(\mathcal{E}_v)$. In analogy with the classical case ($X = H = \mathbb{R}^d$, $A = \mathrm{Id}_H$, $\mu =$ Lebesgue measure) $D(\mathcal{E})$ is a Sobolev space of order one in $L^2(X;\mu)$ which is, therefore, also denoted by $H^{1,2}(X;\mu)$.

Let L denote the generator of the symmetric form $(\mathcal{E}, D(\mathcal{E}))$, resp. L_v that of $(\mathcal{E}_v, D(\mathcal{E}))$, and let $T := e^{tL}$, $S_t := e^{tL_v}$, $t > 0$, denote the corresponding