ALGORITHMIC ORBIT CLASSIFICATION FOR PARABOLIC SUBGROUPS OF SIMPLE ALGEBRAIC GROUPS

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Abstract. For an algebraic group R acting morphically on an algebraic variety X the modality of the action, $\mathrm{mod}(R : X)$, is the maximal number of parameters upon which a family of R-orbits on X depends.

Let G be a reductive algebraic group defined over an algebraically closed field K. Let P be a parabolic subgroup of G. Then P acts on its unipotent radical P_u via conjugation and on $\mathfrak{p}_u$, the Lie algebra of P_u, via the adjoint action. The modality of P is defined as $\mathrm{mod}\, P := \mathrm{mod}(P : \mathfrak{p}_u)$.

In this paper we discuss an algorithm which is used to compute upper bounds for $\mathrm{mod}\, P$ along with some results obtained by this algorithm. One is a classification of parabolic subgroups P of simple algebraic groups G of semisimple rank 2 and modality 0. For parabolic subgroups of semisimple rank 3 we present some partial results. This extends results from [11] and [14] where the cases of semisimple rank 0 and 1 are handled. For exceptional groups G we show that $P \subset G$ has modality zero provided the class of nilpotency of P_u is at most two. The analogous result for classical groups is proved in [20, Theorem 1.3].

For Borel subgroups B of simple groups we are able to determine the value for $\mathrm{mod}\, B$ in some small rank cases by combining lower bounds for $\mathrm{mod}\, B$ from [21] with upper bounds provided by the algorithm.

1. Introduction

Throughout, G is a simple algebraic group over an algebraically closed field K of characteristic $p \geq 0$ and P is a parabolic subgroup of G. Then P acts on its unipotent radical P_u via conjugation and on $\mathfrak{p}_u$, the Lie algebra of P_u, via the adjoint action. The modality of the action of P on $\mathfrak{p}_u$, denoted by $\mathrm{mod}(P : \mathfrak{p}_u)$, is the maximal number of parameters upon which a family of P-orbits on P_u depends; likewise for the action of P on P_u. The modality of P is defined as $\mathrm{mod}\, P := \mathrm{mod}(P : \mathfrak{p}_u)$; see Section 2 for a precise definition.

In this paper we present a number of results which are obtained by an algorithm that is used to calculate upper bounds for $\mathrm{mod}\, P$.

In [14] the general problem was posed of determining each parabolic subgroup P of G with $\mathrm{mod}\, P = 0$. The cases for Borel and semisimple rank one parabolic subgroups were classified in [11] and [14] respectively. We extend these results by classifying parabolic subgroups of G of modality zero and semisimple rank two.

It is shown in [20] for classical groups that $\mathrm{mod}\, P = 0$ provided the nilpotency class of P_u is at most 2. Here we prove the analogous result for exceptional groups.

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With the exception of Lemmas 3.11 and 3.12 the finiteness parts of each result in Section 3 and the upper bounds for $\operatorname{mod} B$ in Section 4 are obtained using our algorithm. We do not furnish additional proofs for any of these cases.

A complete classification of modality zero parabolic subgroups of classical groups is obtained in [10, Theorem 1.1]. Whence we shall focus here especially on modality zero parabolic subgroups of exceptional groups.

For Borel subgroups B of simple groups we determine $\operatorname{mod} B$ in some small rank cases by combining lower bounds for $\operatorname{mod} B$ from [21] with upper bounds provided by the algorithm.

Since the modalities of a linear action and its dual agree (this is an immediate consequence of the proof in [16], e.g., see [12, §2] or [22]), in particular we have $\operatorname{mod}(P : \mathfrak{p}_u) = \operatorname{mod}(P : \mathfrak{p}_u^*)$, where $\mathfrak{p}_u^*$ is the dual of $\mathfrak{p}_u$. Thus we get analogous results for the coadjoint action of P on $\mathfrak{p}_u^*$.

Our program generalizes the algorithm outlined in [7] which was designed to analyze the orbit structure of a Borel subgroup B for the adjoint and coadjoint actions on $\mathfrak{b}_u$ and on $\mathfrak{b}_u^*$. It is based on the usual adjoint action of P on $\mathfrak{p}_u$. The results obtained apply provided $\operatorname{char} K$ does not divide the structure constants of the Chevalley commutator relations for the groups involved. In particular, if the Dynkin-diagram of G is simply laced, no restrictions on $\operatorname{char} K$ are required. For details and various aspects of the algorithm we refer the reader to Section 5 below.

After recalling some definitions and some results from [14], [19], and [22], we present our main results on parabolic subgroups of modality zero in Section 3. In the subsequent chapter we record the values of the modality of Borel subgroups in some simple groups of small rank which were obtained with the aid of this algorithm. Finally, in our last section we outline the algorithm and discuss the various reduction procedures involved in it.

The basic machinery for investigating the modality of parabolic subgroups of reductive groups was introduced in [14]. Apart from [14] itself there are several recent articles related to this subject, such as [6], [9], [10], [13], and [22].

Our general reference for algebraic groups is Borel [3] and for information on root systems we cite Bourbaki [5]. The simple roots in a base of a root system of G are indexed in accordance with *loc. cit.* (Planches I - IX).

2. Recollection, Notation and some Preliminary Results

We denote the Lie algebra of G by $\operatorname{Lie} G$ or by $\mathfrak{g}$, its identity component by G^0 and by e the identity element of G; likewise for subgroups. The Lie algebra of P_u is denoted by $\mathfrak{p}_u$.

Recall from [14] that for an algebraic group R acting morphically on some algebraic variety X the *modality* of this action, is defined as $\operatorname{mod}(R : X) = \max_Z \min_{z \in Z} \operatorname{codim}{}_Z R^0 \cdot z$, where Z runs through all irreducible R^0-invariant subvarieties of X. The invariant $\operatorname{mod}(R : X)$ of this action measures the maximal number of parameters on which a family of R-orbits on X depend upon, and in particular, its value is 0 precisely when R admits only a finite number of orbits on X. The notion of modality originates in the work of Arnold [1]; though our definition here is due to Vinberg [24] and [15, 5.2].

We call $\operatorname{mod} P := \operatorname{mod}(P : \mathfrak{p}_u)$ the *modality* of P, cf. [14].