EXPONENTIAL INTEGRABILITY AND TRANSPORTATION COST RELATED TO LOGARITHMIC SOBOLEV INEQUALITIES *

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Abstract

We study some problems on exponential integrability, concentration of measure, and transportation cost related to logarithmic Sobolev inequalities. On the real line, we then give a characterization of those probablity measures which satisfy these inequalities.

1 Introduction

Logarithmic Sobolev inequalities are an essential tool in the study of many problems (cf. e.g. [G2], [D-SC], [L3]). The main purpose of the present notes is to refine some known connections between logarithmic Sobolev inequalities, exponential integrability of 'smooth' functions and the concentration of measure. Provided a logarithmic Sobolev inequality holds, we estimate exponential moments of functions in terms of the 'modulus of their gradient' and deduce a transportation inequality with a corresponding concentration inequality. Furthemore, we shall describe all probability measures satisfying logarithmic Sobolev inequalities on the real line.

Let us introduce the necessary notations for the general scheme of logarithmic Sobolev inequalities. Let (Ω, μ) denote a probability space, and assume there is an operator Γ defined on a set $\mathcal{A}$ of bounded measurable functions on Ω with the following properties:

1) for any $f \in \mathcal{A}$, $\Gamma(f)$ is a non-negative measurable function on Ω;

2) for all $f \in \mathcal{A}$ and for all $a \in \mathbf{R}$, $b \geq 0$, $a + bf \in \mathcal{A}$ and $\Gamma(a + bf) = b\,\Gamma(f)$.

Introduce the entropy functional:

$$\begin{aligned} \mathrm{Ent}\,(g) \;&\equiv\; \int g \log g \, d\mu - \int g \, d\mu \, \log \int g \, d\mu \\ &= \int g \, \log \frac{g}{\int g \, d\mu} \, d\mu, \qquad g \geq 0. \end{aligned}$$

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We will say that (Ω, μ, Γ) satisfies a logarithmic Sobolev inequality with constant $c \geq 0$ (for short, $\mathrm{LSI_c}$) if, for all $f \in \mathcal{A}$,

$$\mathrm{Ent}\,(e^f) \ \leq \ \frac{c}{2} \int \Gamma(f)^2 e^f \, d\mu. \tag{1.1}$$

If Γ is a derivation, i.e., if for all $f \in \mathcal{A}$ with values in (a, b) and $u \in C^\infty[a, b]$, we have $u(f) \in \mathcal{A}$ and $\Gamma(u(f)) = |u'(f)|\,\Gamma(f)$, then (1.1) is equivalent to the more familiar version

$$\mathrm{Ent}\,(g^2) \ \leq \ 2c \int \Gamma(g)^2 \, d\mu, \tag{1.2}$$

whenever $g \in \mathcal{A}$ satisfies $\inf\, g > 0$. To be more precise, (1.1) represents an exponential (or, modified) form of (1.2) to which one ordinarily referes as the 'true' logarithmic Sobolev inequality (the normalization $\frac{1}{2}$ in (1.1) is chosen in order to have $c = 1$ in the Gaussian case). In most interesting cases of derivations Γ, this operator and the inequality (1.2) itself (as well as (1.1)) are easily extended to a larger class of functions including unbounded 'smooth' functions. For example, when Ω is a metric space with metric d, one may consider the following natural generalization of the modulus of the usual gradient:

$$\Gamma(f)\,(x) = |\nabla f(x)| = \limsup_{d(x,y) \to 0^+} \frac{|f(x) - f(y)|}{d(x, y)}.$$

In this case, we may define Γ on the class of all Lipschitz functions f (i.e., with $\|f\|_{\mathrm{Lip}} < \infty$), and the inequalities (1.1) and (1.2) hold for this class if they hold for all bounded Lipschitz functions.

In section 2 we study exponential moments of a function $f \in \mathcal{A}$ in terms of the distribution of $\Gamma(f)$ provided $\mathrm{LSI_c}$ holds. As a main result (cf. Theorem 2.1), we prove that, for any function $f \in \mathcal{A}$ with mean $\int f \, d\mu = 0$ and for all $\alpha > c/2$,

$$\int e^f \, d\mu \ \leq \ \left(\int e^{\alpha\, \Gamma(f)^2} \, d\mu \right)^{c/(2\alpha - c)}. \tag{1.3}$$

This estimate implies some inequalities for exponential moments of 'Lipschitz' functions similar to the ones due to S.Aida, T.Masuda and I.Shigekawa [A-M-S] and M.Ledoux [L1], [L2]. In the Gaussian case, (1.3) with $\alpha = c = 1$ improves an exponential inequality due to G.Pisier [P1]. In the case of the discrete cube, it affirmatively answers his question (p.182) on the validity of the discrete analogue of the Gaussian variant of (1.3). These applications will be discussed in more detail in section 2.

In section 3, we study, for a metric space (Ω, d) equipped with a probability measure μ, inequalities of the following type:

$$W_\alpha(\mu, \nu) \ \leq \ \sqrt{2c \int \log \frac{d\nu}{d\mu} \, d\nu}. \tag{1.4}$$

Here $W_\alpha(\mu, \nu)$ denotes the Kantorovich–Rubinstein distance between μ and ν (we shall consider the cases $\alpha = 1$ and $\alpha = 2$, only) which is defined as the infimum of

$$\left(\iint d(x, y)^\alpha \, d\pi(x, y) \right)^{1/\alpha}$$