Cellular algebras: inflations and Morita equivalences

by

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1 Introduction

Cellular algebras have been introduced recently by Graham and Lehrer [5, 6] as a convenient axiomatization of all of the following algebras, each of them containing information on certain classical algebraic or finite groups: group algebras of symmetric groups in any characteristic, Hecke algebras of type A or B (or more generally, Ariki Koike algebras), Brauer algebras, Temperley–Lieb algebras, (q–)Schur algebras, ... The problem to determine a parameter set for or even to construct bases of simple modules is in this way reduced (but of course not solved in general) to questions of linear algebra.

The present paper has two aims. First, we make explicit an inductive construction of cellular algebras which has as input data of linear algebra and which produces in fact all cellular algebras (but no other ones). This is what we call 'inflation'. This construction also exhibits close relations between several of the above algebras as can be seen from the computations in [6]. Among the consequences of the construction is a natural way to generalize Hochschild cohomology. Another consequence is a construction of certain idempotents which is used in the second part of the paper.

The second aim is to study Morita equivalences of cellular algebras. Since the input of many constructions of representation theory of finite dimensional algebras is a basic algebra, it is useful to know whether any finite dimensional cellular algebra is Morita equivalent to a basic one by a Morita equivalence preserving the cellular structure. It turns out that the answer is yes if the underlying field has characteristic different from two. But there are counterexamples in case of characteristic two, or more generally for any ring in which 2 is not invertible. This also tells us that the notion of 'cellular' cannot be defined in terms of the module category only. However, in any characteristic we find some useful Morita equivalences which are compatible with cellular structures.

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In more detail, the contents of this paper is as follows: In section two we recall two equivalent definitions of cellular. In section three we define inflations, and in section four we explain how to produce cellular algebras by inflations. Section five illustrates this construction by examples from [6]. In section six we look at Morita equivalences which are compatible with given involutory anti–automorphisms, and we prove some technical results about the existence of such Morita equivalences. In particular, proposition 6.10 gives a sufficient criterion for a Morita equivalence to send a cell ideal to a cell ideal. More convenient for practical purposes is another sufficient criterion, given in corollary 6.12. In sections seven and eight we answer the question of whether any cellular algebra is Morita equivalent to a basic one in a way which preserves the cellular structure. Section seven is negative: it contains our counterexample in case of characteristic two and also an example of a full matrix algebra which is not cellular with respect to a certain involution. Section eight is positive; here we give the main result in case of characteristic different from two (using the techniques developed in section six). Among the consequences of the main result is that over fields of characteristic different from two any endomorphism ring of a projective module over a cellular algebra inherits the cellular structure of the algebra. Moreover, the maximal semisimple quotient of a cellular algebra always inherits the cellular structure (in any characteristic).

2 Definitions

Here we recall the original definition (due to Graham and Lehrer) and our equivalent definition [8] which we are going to use later on.

In the following by an A–module we mean a finitely generated A–module. By $A-mod$ or $mod-A$ we denote the category of finitely generated left or right A–modules, respectively.

The definition by Graham and Lehrer is as follows.

Definition 2.1 *(Graham and Lehrer, [6]) Let R be a noetherian commutative integral domain. An associative R–algebra A is called a* **cellular algebra** *with cell datum (I, M, C, i) if the following conditions are satisfied:*

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has an R–basis $C_{S,T}^{\lambda}$ where (S,T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is an R–linear anti–automorphism of A which satisfies $i^2 = id$ and which sends $C_{S,T}^{\lambda}$ to $C_{T,S}^{\lambda}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC_{S,T}^{\lambda}$ can be written as $(\sum_{U \in M(\lambda)} r_a(U,S) C_{U,T}^{\lambda}) + r'$ where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U,S) \in R$ do not depend on T.

In the following, we call the basis $\{C_{S,T}^{\lambda}\}$ a **cellular basis** for the cellular algebra A, and an R–linear anti–automorphism i of A with $i^2 = id$ an **involution**. By $J_{\leq\lambda}$ (or