On subgroups of prime power index

Barbara Baumeister

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Abstract

We determine all finite groups G which admit a subgroup K of index p^a, p a prime, under the assumption that G has an irreducible and faithful $GF(p)$-module of dimension at most a. As an application to the theory of permutation groups we determine the maximal transitive subgroups of the primitive affine permutation groups.

1 Introduction.

In [Gu] Guralnick classified all the finite simple groups which possess a subgroup of prime power index. In this paper we determine all the finite groups G which admit a subgroup K of index p^a, p a prime, under the assumption that G has an irreducible and faithful $GF(p)$-module of dimension at most a.

Our main theorem is as follows.

Theorem 1 *Suppose that G is a finite group which contains a subgroup K of index p^a such that a is at least the dimension d of an irreducible and faithful $GF(p)$-modul V of G. Then the following holds.*

(i) $F(G) = 1$;

(ii) $p = 2$ *and* $d = 3r$, *for an* $r \in \mathbb{N}$;

(iii) $E(G) = T_1 \times \ldots \times T_r$, *where* $T_i \cong L_3(2)$, *for* $i = 1, \ldots, r$;

(iv) $G \cong T_1 \wr X$, *where X is a transitive subgroup of* S_r;

(v) $E(G) \not\leq K$ *and either* $T_i \leq K$ *or* $K \cap T_i \cong \mathrm{Frob}(7:3)$, *for* $i = 1, \ldots, r$;

(vi) $V = [V, T_1] \oplus \ldots \oplus [V, T_r]$, *where $[V, T_i]$ is the natural module for T_i.*

Theorem 1 generalizes the following more specialized version proved in [Ba1].

Theorem 2 *[Ba1, Proposition (2.7.1)] Suppose that G is as in Theorem 1 and that $F(G) = 1$. Then (ii) – (vi) of Theorem 1 hold.*

We consider the following hypothesis (A)

(A) The group G has a subgroup K of index p^a, p a prime, and G has an irreducible and faithful $GF(p)$-module V of dimension d, where d is at most a.

Remark 1. In fact it can happen that $T_i \leq K$, for some $i \in \{1, \ldots, r\}$. The following group provides us with such an example.

Let $T_1, \ldots, T_8$ be copies of $L_3(2)$, take $S \cong L_2(7)$ in its action of degree 8 and let G be the wreath product

$$G = (T_1 \times \cdots \times T_8) : S = T_1 \wr S.$$

As described in Theorem 1(vi), G has an irreducible and faithful $GF(2)$–module of dimension 24. Let K_2 be a subgroup of T_2 with $K_2 \cong \text{Frob}(7:3)$ and let S_1 be the stabilizer of T_1 in S. Then $S_1 \cong \text{Frob}(7:3)$ and $\langle K_2^{S_1} \rangle \cong \text{Frob}(7:3)^7$. Set $K = T_1 \times \langle K_2, S_1 \rangle$. Then K has index $|G:K| = 2^{24}$. Thus G fulfils property (A), but T_1 is contained in K.

Let (G, Ω) be a primitive group of affine type, that is $G = O_p(G)G_w$ with G_w the stabilisator of $w \in \Omega$ and $O_p(G)$ an irreducible and faithful $GF(p)G_w$-module. The initial motivation for Theorem 1 was the wish to prove the following theorem.

Theorem 3 *Let (G, Ω) be a primitive permutation group of affine type. Suppose that B is a maximal subgroup of G which acts transitively on Ω. Then one of the following hold.*

(i) $O_p(G) \leq B$;

(ii) $O_p(G)$ *is elementary abelian of order 2^{3r} for some positive integer r, $G \cong (E_8 : L_3(2)) \wr X$ for some transitive subgroup X of S_r and $B \cong L_3(2) \wr X$.*

This theorem generalizes for the affine case the classification given in [Ba2]. There all the finite primitive permutation groups (G, Ω) which admit a factorization $G = G_\omega G_\omega^\alpha$ with $\alpha \in Aut(G)$ are determined.

The paper is organized as follows: In the next section we prove two lemmas on the order of the outer automorphism group of a simple group and in the third section we recall the theorem of Clifford and give simple estimations of the dimension of some special $GF(p)G$-modules. We will need the results of the two sections in the proof of Theorem 1. In the fourth section we prove, if a group G, which has a normal p-complement, but no normal p-subgroup, has a faithful $GF(p)$-module V of dimension d, then any Sylow p-subgroup of G has less then p^d elements. In Section 5 we prove Theorem 1 and in Section 6 Theorem 3.

The idea of the proof of Theorem 1 is as follows: Using the results of sections 2 through 4 we give bounds for a and d. These estimates imply that there is a component T of G satisfying the following property: