Biextensions by indecomposable modules of derived regular length 2

Peter Dräxler and Andrzej Skowroński

1 Introduction and main results

In this paper 'algebra' means a connected, basic, finite-dimensional algebra (associative, with 1) over an algebraically closed field k. For an algebra A we denote by $A-\mathrm{mod}$ the category of finite-dimensional left A-modules and by $D^b(A)$ the derived category of bounded complexes over $A-\mathrm{mod}$.

By Drozd's 'Tame and Wild Theorem' ([Dd], see also [CB1] and [G-V]) the class of algebras may be divided into two disjoint parts. Firstly, there are the tame algebras for which the indecomposable modules occur, in each dimension d, up to isomorphism in a finite number of discrete and a finite number of one-parameter families. Secondly, there are the wild algebras whose representation theory is at least as complicated as the study of finite-dimensional vector spaces with two non-commuting endomorphisms, for which the classification up to isomorphism is a well-known unsolved problem.

We are concerned with the problem of deciding when a given algebra is tame. Frequently, using deformations and coverings, we may reduce it to the tameness problem for algebras whose ordinary quiver is directed. For this class of algebras we may often solve it by starting with known classes of tame algebras and applying iterated one-point extensions and coextensions.

At present, the most developed is the representation theory of tame algebras of polynomial growth (see [GP], [Sk] for surveys and more references). In this theory an important role is played by the representation-infinite tilted algebras of Euclidean type and their suitable enlargements ([AS], [AST], [Ri2]). The representation theory of tame non-polynomial growth algebras is rather poor and only few classes of algebras are understood ([BR], [CB2], [DS], [Er], [WW]).

In [Ri1] it is shown that, if H is a tame hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_n$ and R is an indecomposable regular H-module of regular length 2 in a tube of rank $n-2$, then the one-point extension $H[R]$ is tame but not of polynomial growth. In general, it is open what happens if one considers the multiple one-point extension $H[R_1, \dots, R_t] := H[R_1]\dots[R_t]$ where the R_i are H-modules like R above. The reason is that if one wants to work by induction, then the module categories of the intermediate algebras $A[R_1, \dots, R_i]$ are not well-understood and consequently the usual one-point extension technique based on the calculation of the vector space category $(H[R_1, \dots, R_i]-\mathrm{mod}, \mathrm{Hom}_{H[R_1,\dots,R_i]}(R_{i+1}, -))$ fails.

The aim of this paper is to solve more general problems. In our approach we apply derived categories as introduced in [Ha] and recent results on fiber sum functors and generalized one-point extensions proved by the first named author in [Dr1], [Dr3].

As preparation we need to introduce a generalization of one-point extension and coextension algebras. For this purpose let $R'_1, \dots, R'_s$ and $R_1, \dots, R_t$ be two sequences of modules over an algebra B. We put $R' := \oplus_{i=1}^s R'_i$ which we consider as an B-k^s-bimodule. Analogously we consider $R := \oplus_{i=1}^t R_i$ as an B-k^t-bimodule. The *biextension algebra* $[R'_1, \dots, R'_s]B[R_1, \dots, R_t]$ of B by the two sequences $R'_1, \dots, R'_s$ and $R_1, \dots, R_t$ is by definition the matrix algebra

$$\begin{pmatrix} k^t & 0 & 0 \\ R & B & 0 \\ \mathrm{D}(R') \otimes_B R & \mathrm{D}(R') & k^s \end{pmatrix}$$

equipped with the obvious addition and multiplication. For $s = 0$ (resp. $t = 0$) one obtains the usual iterated one-point extension $B[R_1, \dots, R_t]$ (resp. iterated one-point coextension $[R'_1, \dots, R'_s]B$). If E is the right adjoint of the canonical restriction functor $B[R_1, \dots, R_t]-\mathrm{mod} \to B-\mathrm{mod}$ and L is the left adjoint of the canonical restriction functor $[R'_1, \dots, R'_s]B-\mathrm{mod} \to B-\mathrm{mod}$, then $[R'_1, \dots, R'_s]B[R_1, \dots, R_t] = [E(R'_1), \dots, E(R'_s)](B[R_1, \dots, R_t]) = ([R'_1, \dots, R'_s]B)[L(R_1), \dots, L(R_t)]$.

We shall describe now the main results of the paper. Let $\vec{\Delta}$ be a finite, directed, connected quiver with underlying graph Δ. Following [Ha] an algebra B is said to be piecewise hereditary of type $\vec{\Delta}$ if there is a triangle equivalence $F : D^b(B) \to D^b(H)$ where H is the finite-dimensional hereditary algebra $k\vec{\Delta}$. For Δ an Euclidean graph the structure of $D^b(H)$ is known rather precisely. Namely, the Auslander-Reiten quiver consists of a sequence $\mathcal{P}[\nu]$, $\nu \in \mathbb{Z}$, of directed components of type $\mathbb{Z}\vec{\Delta}$ and a sequence $\mathcal{R}[\nu]$, $\nu \in \mathbb{Z}$, of families of stable tubes of tubular type Δ^0 where Δ^0 is the Dynkin diagram attached naturally to Δ. By abuse of language, we also denote by $\mathcal{P}[\nu]$ and $\mathcal{R}[\nu]$ the corresponding full subcategories of $D^b(H)$ which are known to be standard. Note that the $\mathcal{R}[\nu]$ are abelian length categories.

After embedding $B-\mathrm{mod}$ into $D^b(B)$ in the usual way, we call *derived regular* just those modules $X \in B-\mathrm{mod}$ such that $F(X)$ lies in one of the subcategories $\mathcal{R}[\nu]$. For a derived regular module X we denote its length as object of $\mathcal{R}[\nu]$ as *derived regular length* of X.

We will denote by $\tau_{D^b(H)}$ the Auslander-Reiten translation in $D^b(H)$ and by T the shift functor. Using τ, we will usually skip the index. The $\mathcal{P}[\nu]$ and $\mathcal{R}[\nu]$ are arranged to satisfy $T\mathcal{P}[\nu] = \mathcal{P}[\nu+1]$ and $T\mathcal{R}[\nu] = \mathcal{R}[\nu+1]$.

Two objects X, Y of an additive category are said to be *Hom-orthogonal* if there is no non-zero map from X to Y and from Y to X. With these notations our first theorem is the following:

Theorem A. *Let B be a piecewise hereditary algebra of type $\vec{\Delta}$ where Δ is $\widetilde{\mathbb{A}}_n$ or $\widetilde{\mathbb{D}}_n$. Suppose $\{R'_1, \dots, R'_s\}$ and $\{R_1, \dots, R_t\}$ are sets of indecomposable derived regular B-modules of derived regular length 2 such that their images under F lie in non-homogeneous*