On the Zariski closure of the linear part of a properly discontinuous group of affine transformations

by

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1. Introduction

A well known classical theorem due to Bieberbach says that every discrete group Γ of isometries of the n—dimensional Euclidean space $\mathbb{R}^n$ with compact quotient $\Gamma \setminus \mathbb{R}^n$ contains a subgroup of finite index consisting of translations. Hence such a group Γ is virtually abelian, i.e. Γ contains an abelian subgroup of finite index.

Let us now look at the group of affine transformations instead of the group of isometries of $\mathbb{R}^n$. More precisely, let $G_n = \mathrm{Aff}(\mathbb{R}^n)$ denote the group of affine transformations of $\mathbb{R}^n$. The group G_n is the semidirect product $GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$ where $\mathbb{R}^n$ is identified with its group of translations. A subgroup Γ of G_n is said to act *properly discontinuously* on $\mathbb{R}^n$ if for every compact subset K of $\mathbb{R}^n$ the set $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$ of recurrencies is finite. If a discrete group Γ consists of isometries then Γ acts properly on $\mathbb{R}^n$. But this is not true for all discrete subgroups of G_n, e.g. for an infinite discrete subgroup of $GL_n(\mathbb{R})$.

A subgroup Γ of G_n will be called *crystallographic* if Γ acts properly discontinuously on $\mathbb{R}^n$ and the orbit space $\Gamma \setminus \mathbb{R}^n$ is compact. In [A] Auslander conjectured that every crystallographic subgroup Γ of G_n is virtually solvable, i.e. contains a solvable subgroup of finite index. In [Mi] Milnor asked whether Auslander's conjecture is true without the assumption that the orbit space $\Gamma \setminus \mathbb{R}^n$ be compact.

Milnor's question has a positive answer for $n \leq 2$ (easy for $n = 2$, trivial for $n = 1$). But for $n = 3$ the answer to Milnor's question is negative. In fact, the second named author proved that there is a nonabelian free subgroup Γ of $\mathrm{Aff}(\mathbb{R}^3)$ acting properly discontinuously on $\mathbb{R}^3$ ([Ma 1,2]). On the other hand, D. Fried and W. Goldman [FG] proved Auslander's conjecture for $n = 3$ using cohomological arguments. For higher dimensions the existing results confirming the Auslander conjecture are proved under the assumption that the linear part $l(\Gamma)$ of Γ belongs to some special subgroup of $GL_n(\mathbb{R})$ where $l : G_n \to GL_n(\mathbb{R})$ denotes the natural homomorphism ([GK, GrM, S, To]). The most general result in this direction was obtained by Soifer and Tomanov ([S, To]), as follows.

Theorem 1 *Let Γ be a crystallographic subgroup of G_n. Suppose Γ is a group of generalized Lorentz motions, i.e. $l(\Gamma)$ belongs to an algebraic subgroup H of $GL_n(\mathbb{R})$ such that the semisimple quotient $S = H^\circ/R$ of the connected component H° by its solvable radical R is of the form $S = S_1 \ldots S_r$ where the S_i are the simple factors of S and $\mathrm{rank}_{\mathbb{R}} S_i \leq 1$ for $1 \leq i \leq r$. Then Γ is virtually solvable.*

The two main results of the present paper are

Theorem 2 *For n even there is no properly discontinuous subgroup Γ of G_{2n+1} with linear part $l(\Gamma)$ Zariski dense in $SO(n+1, n)$.*

Theorem 3 *For n odd there are properly discontinuous free subgroups Γ of G_{2n+1} with linear part $l(\Gamma)$ Zariski dense in $SO(n+1, n)$.*

The Auslander conjecture in dimensions $n \leq 5$ follows from theorem 2, theorem 1 and other results of [To] and results announced in [Ma 3].

Theorem 2, its proof and the statement of theorem 3 were contained in the preprint [Ma 3] of Margulis circulated in 1991. The results of the present paper and more were announced in [AMS 2]

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