Invariants of 2×2 matrices

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Studying invariants of a set of 2×2 matrices is important for the theory of Kleinian and Fuchsian groups. There are many papers treating this subject (see, e.g., [FK], [Si], [He1], [Ho], [Ra], [Pr], [Tr], [LM], [Le], [Re], [Vi2], [VHM], [He2] and references there). The present paper has the following aims:

1) to systematize the results on invariants of 2×2 matrices;
2) to present them in a convenient form;
3) to show that they are easy consequences of the classical description of orthogonal invariants of vectors and standard facts of geometric invariant theory.

Some minor observations may be considered new.

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1. Statement of the main theorems.

We consider the action of the group $G = SL_2(\mathbb{C})$ on $G^n := \underbrace{G \times \cdots \times G}_{n}$ by simultaneous conjugations:

$$g \circ (X_1, \ldots, X_n) = (gX_1g^{-1}, \ldots, gX_ng^{-1}).$$

We put $G^* = G \times C_2$, where C_2 is a cyclic group of order 2 with generating element ι . The action of G is extended to an action of G^* by

$$\iota \circ (X_1, \ldots, X_n) = (X_1^{-1}, \ldots, X_n^{-1}).$$

For any set

$$\mathbf{X} = (X_1, \ldots, X_n) \in G^n$$

we put

$$t_{ij}(\mathbf{X}) = t_{ij} = \operatorname{tr} X_i X_j^{-1} \ (i, j = 0, 1, \ldots, n),$$

where $X_0 = E$. In particular, we have

$$t_{ii} = 2, \quad t_{ij} = t_{ji}. \tag{1}$$

The $(n+1) \times (n+1)$ matrix

$$T(\mathbf{X}) = T = (t_{ij})$$

is called *the trace matrix* of $\mathbf{X}$.

Theorem 1 *The algebra* $\mathbb{C}[G^n]^{G^*}$ *of* G^**–invariant polynomial functions on* G^n *is generated by* t_{ij} *'s, with defining relations* (1) *and*

$$\operatorname{rk} T \leq 4. \tag{2}$$

(The latter relations mean that all the minors of order 5 of T equal 0. In fact, since $t_{00} = 2$, one may restrict oneself with the minors containing t_{00}).

Consider *the factorization morphism*

$$\pi : G^n \to S, \quad \mathbf{X} \longmapsto T(\mathbf{X}),$$

where S denotes the variety of $(n+1) \times (n+1)$ matrices satisfying (1) and (2).

Theorem 2 *The morphism* π *is surjective. Its fiber* $\pi^{-1}(T)$ *over* $T \in S$ *is described as follows:*

1) *if* $\operatorname{rk} T = 4$ *, the fiber consists of a single 3–dimensional* G^**–orbit, which decomposes into two* G*–orbits;*

2) *if* $\operatorname{rk} T = 3$ *, the fiber consists of a single 3–dimensional* G^**–orbit, which is one* G*–orbit;*

3) *if* $\operatorname{rk} T = 2$ *, the fiber consists of a 2–dimensional* G^**–orbit, which is one* G*–orbit, and an* $(n-2)$*–parameter family of 3–dimensionl* G^**–orbits, each of them decomposes into two* G*–orbits (if* $n = 2$ *, the family consists of a single orbit; if* $n = 1$ *, it is empty);*