Primitive and Quasi Primitive Elements in Weyl Modules for Type A_2

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The submodule structure of Weyl modules for SL_3 has been determined in [DS, K, I] and by Cline (unpublished). The main technique used in [DS] is an analysis of the interwining homomorphisms between certain Weyl modules. The approach in [I, K] is essentially a direct calculation. Also Jantzen's translation principal plays an important role in these works. For a few special cases, Irving gives some (quasi) primitive elements in Verma modules of the Frobenius kernel and Weyl modules (see [I]). In this paper we will find out all (quasi) primitive elements in Verma modules of the quantized Frobenius kernel and Weyl modules for type A_2, see §§2-3. Thus we understand the structure of the Verma modules and the Weyl modules more explicitly. Our method is based on the main results in [X1] and is rather simple. In this paper we only work with quantized enveloping algebras at roots of 1 (Lusztig version), for hyperalgebras the approach is completely similar. The method works for type B_2 (see [X2]). The author also did some computations for type A_3 by using the method, although the amount of the computation is large, it seems still not difficult to work out the structure of Weyl modules for type A_3.

1. Primitive and Quasi Primitive Elements

In this section we fix notation and give the definition for (quasi) primitive elements. We refer to [L1-4, X1] for additional information.

1.1. Let U_ξ be a quantized enveloping algebra (over $\mathbf{Q}(\xi)$) at a root ξ of 1. We assume that the order of $\xi \geq 3$. As usual, the generators of U_ξ are denoted by $E_i^{(a)}, F_i^{(a)}, K_i, K_i^{-1}$, etc. Let $\mathbf{u}$ be the Frobenius kernel and $\tilde{\mathbf{u}}$ the subalgebra of U_ξ generated by all elements in $\mathbf{u}$ and in the zero part of U_ξ. For $\lambda \in \mathbf{Z}^n$ and a U_ξ-module or a $\tilde{\mathbf{u}}$-module M we denote by M_λ the λ-weight space of M. An element in M_λ will be called an element of weight λ or a weight vector. We call m primitive if m is a nonzero weight vector and $E_i^{(a)}m = 0$

for all i and $a \geq 1$. Let $m \in M$ be a weight vector. If there exist two submodules M_1, M_2 of M such that the image in M_1/M_2 of m is primitive, then we call m a quasi primitive element. Obviously, primitive elements are quasi primitive. We have:

(a) Let $m \in M$ be a weight vector and let M_1 be the submodule of M generated by m and let N be a maximal submodule of M. Then m is quasi primitive if and only if the image in M/N of m is primitive.

(b) If m is a quasi primitive element of weight λ, then $L(\lambda)$ (or $\tilde{L}(\lambda)$) is a composition factor of M (depending on M is a U_ξ-module or a $\tilde{\mathbf{u}}$-module). Here and later we write $L(\lambda)$ (resp. $\tilde{L}(\lambda)$) for an irreducible U_ξ-module (resp. $\tilde{\mathbf{u}}$-module) of highest weight λ.

(c) Let M and N be modules and $\phi : M \to N$ a homomorphism. Let m be a weight vector in M. If $\phi(m)$ is a quasi primitive element of N, then m is a quasi primitive element of M.

(d) Let M, N, ϕ, m be as in (iii). If m is a (quasi) primitive element of M and $\phi(m) \neq 0$, then $\phi(m)$ is a (quasi) primitive element of N.

We shall denote by $\tilde{Z}(\lambda)$ the Verma module of $\tilde{\mathbf{u}}$ with highest weight λ and denote by $\tilde{1}_\lambda$ a nonzero element in $\tilde{Z}(\lambda)_\lambda$. Recall that to define U_ξ we need to choose $a_i \in \{1, 2, 3\}$ such that $(a_i a_{ij})$ is symmetric, where (a_{ij}) is the concerned Cartan matrix. Assume the rank of the Cartan matrix is n. Let l_i be the order of ξ^{2d_i}. For $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbf{Z}^n$ we set $\mathbf{l}\lambda = (l_1\lambda_1, ..., l_n\lambda_n)$. We call λ is $\mathbf{l}$-restricted if $0 \leq \lambda_i \leq l_i - 1$ for all i. Denote by $\mathbf{Z}^n_{+,\mathbf{l}}$ the set of all $\mathbf{l}$-restricted elements in $\mathbf{Z}^n$.

(e) Let $\lambda' \in \mathbf{Z}^n_{+,\mathbf{l}}$, $\lambda'' \in \mathbf{Z}^n$. Set $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^n$. Then $F_i^{(\lambda'_i+1)}\tilde{1}_\lambda$ is primitive if $\lambda'_i \neq l_i - 1$.

2. Structure of $\tilde{Z}(\lambda)$ for Type A_2

2.1. From now on we assume that U_ξ is of type A_2. In this section we determine the (quasi) primitve elements in $\tilde{Z}(\lambda)$ (or equivalently in any highest weight module of $\tilde{\mathbf{u}}$) and the submodule structure of $\tilde{Z}(\lambda)$, see 2.2-7. We choose $d_i = 1$ and let $l' = l_1 = l_2$. The following result is a special case of [X1, 4.2 (ii)].