HOMOMORPHISMS BETWEEN REPRESENTATIONS OF CLANS

CHRISTOF GEISS

Abstract. We give a description of the homomorphisms between representations of a clan [CB2], more or less as it was done in [Kr] for string-algebras. As a consequence of our findings we can establish the irreducibility of all maps which are expected to be so. This allows us, after defining a natural exact structure on the category of representations, to calculate in many situations Auslander-Reiten sequences like in the string-algebra case.

Introduction

In [CB2] Crawley-Boevey introduced clans, a class of problems which includes several well-known problems from representation theory, like the Gelfand-problem [Gf], self-reproducing systems [NRo] or semi-dihedral algebras [CB3]. In [CB2] Crawley-Boevey gives also a complete classification of the indecomposable representations of such a clan, similar to the well known classification of indecomposable modules for string-algebras (see for example [BR]), basically by certain words. The original description of the indecomposable representations can be improved by the work of Bondarenko [Bd] and Deng [De].

For string algebras there is moreover a well established description of the homomorphisms between representations, see [CB4] and [Kr], as well as a handy description of Auslander-Reiten sequences. In the present paper we supply analogous results for representations of clans and clannish algebras.

In section 1 we recall the definition of a clan $\mathcal{C}$ and of the category of its representations $\mathsf{R}(\mathcal{C})$, and introduce by the way some conventions about (bi-) quivers, words and letters.

Section 2 is dedicated to the description of the indecomposable representations of the clan $\mathcal{C}$. We follow essentially [CB2], though the proof that this particularly nice description works is due to [Bd] or [De]. We define a set $\mathfrak{X}$ of words, and functors $S_{\mathfrak{w}} : A_{\mathfrak{w}}\text{-}\mathsf{mod} \to \mathsf{R}(\mathcal{C})$ which parameterize the indecomposable representations. An important feature of this description is, that the functors $S_{\mathfrak{w}}$ are symbolized by certain almost linear quivers $H^{\mathfrak{w}}$ for $\mathfrak{w} \in \mathfrak{X}$ plus some extra data.

Date: 04.09.97.

1991 *Mathematics Subject Classification.* 16G70.

Key words and phrases. Clans, homomorphisms, Auslander-Reiten sequences.

In section 3 we present the system of linear equations to solve for $\mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$, codify this data in a quiver $H^{\mathfrak{w},\mathfrak{v}}$ (see 3.3) and state several auxiliary combinatorial facts. In section 4 we present and prove our findings, in fact it turns out, that the morphisms in $\mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$ can be read off conveniently from the quiver $H^{\mathfrak{w},\mathfrak{v}}$ (see 3.3) plus some extra data. Section 5 is dedicated to Auslander-Reiten theory: Despite the fact, that in case of asymmetric bands we can not write down an explicit base, we have enough information to prove that that all "candidates" for irreducible morphisms are in fact irreducible. After introducing a natural exact structure on $\mathsf{R}(\mathcal{C})$ we present canonical exact sequences involving the irreducible maps, and obtain thus the existence of Auslander-Reiten sequences under reasonable hypothesis on $\mathcal{C}$.

The remaining two sections contain the proofs of the more technical results from 3, linear algebra in 6 and combinatorics in 7.

I thank M. Barot and R. Bautista for helpful discussions.

1. Basic constructions

1.1. **Quivers and letters.** A *biquiver* Q is a tuple (Q_0, Q_1, Q_2, s, t) with Q_0 the vertices, Q_1 the arrows, Q_2 the special arrows, and $s, t\colon (Q_1 \cup Q_2) \to Q_0$ defining the start- and terminal point of the arrows, thus $s(a) \xrightarrow{a} t(a)$ for each arrow $a \in Q_1$ and $s(b) \overset{b}{\dashrightarrow} t(b)$ for each $b \in Q_2$.

For technical reasons we need frequently a *polarization* of Q, i.e. a pair of functions $\sigma, \tau\colon (Q_1 \cup Q_2) \to \{-1, 1\} \subset \mathbb{Z}$.

The set of *letters* Q_* of the quiver Q consists of the following three types of symbols l

type of l	symbol l		$s(l)$	$t(l)$	$\sigma(l)$	$\tau(l)$
direct	$\underline{\alpha}$	$\alpha \in Q_1$	$s(\alpha)$	$t(\alpha)$	$\sigma(\alpha)$	$\tau(\alpha)$
inverse	$\underline{\beta}^{-1}$	$\beta \in Q_1$	$t(\beta)$	$s(\beta)$	$\tau(\beta)$	$\sigma(\beta)$
special	$\underline{\varepsilon}^*$	$\varepsilon \in Q_2$	$s(\varepsilon)$	$t(\varepsilon)$	$\sigma(\varepsilon)$	$\tau(\varepsilon)$

We have also indicated the definition of the functions s, t, σ and τ for the letters.

1.2. **Definition.** A *clan* $\mathcal{C}$ is a tuple $(\mathrm{k}, Q, (\sigma, \tau) \geq)$ where k is a field, Q a biquiver with polarization (σ, τ), such that $s(\varepsilon) = t(\varepsilon)$ and $\sigma(\varepsilon) = \tau(\varepsilon)$ for all $\varepsilon \in Q_2$, and $\geq$ is a linear order on each set of letters

$$Q_*(p, \rho) := \{l \in Q_* \mid t(l) = p, \tau(l) = \rho\}.$$

A *representation* M of $\mathcal{C}$ consists of the following data:

(R1) a k-vector-space M_p for each $p \in Q_0$,