(NONSYMMETRIC) DIRICHLET OPERATORS ON L^1: EXISTENCE, UNIQUENESS AND ASSOCIATED MARKOV PROCESSES

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Abstract. Let L be a nonsymmetric operator of type $Lu = \sum a_{ij}\partial_i\partial_j u + \sum b_i\partial_i u$ on an arbitrary subset $U \subset \mathbb{R}^d$. We analyse L as an operator on $L^1(U,\mu)$ where μ is an invariant measure, i.e., a possibly infinite measure μ satisfying the equation $L^*\mu = 0$ (in the weak sense). We explicitely construct, under mild regularity assumptions, extensions of L generating sub–Markovian C_0–semigroups on $L^1(U,\mu)$ as well as associated diffusion processes. We give sufficient conditions on the coefficients so that there exists only one extension of L generating a C_0–semigroup and apply the results to prove uniqueness of the invariant measure μ. Our results imply in particular that if $\varphi \in H^{1,2}_{loc}(R^d, dx)$, $\varphi \neq 0$ dx–a.e., the generalized Schrödinger operator $(\Delta + 2\varphi^{-1}\nabla\varphi\cdot\nabla, C_0^\infty(R^d))$ has exactly one extension generating a C_0–semigroup if and only if the Friedrich's extension is conservative. We also give existence and uniqueness results for a corresponding class of infinite dimensional operators acting on smooth cylinder functions on a separable real Banach space.

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0. Introduction

The purpose of this paper is to study the Cauchy–Problem of second order differential operators with measurable coefficients of type

$$Lu = \sum_{i,j=1}^{d} a_{ij}\partial_i\partial_j u + \sum_{i=1}^{d} b_i\partial_i u \tag{0.1}$$

defined on $C_0^\infty(U)$, $U \subset \mathbb{R}^d$ open, on the space $L^1(U,\mu)$, where μ is an *invariant measure*, i.e., a possibly infinite measure satisfying $Lu \in L^1(U,\mu)$ for all $u \in C_0^\infty(U)$ and

$$\int Lu\,d\mu = 0 \text{ for all } u \in C_0^\infty(U)\ . \tag{0.2}$$

Here the $(a_{ij})_{1\le i,j\le d}$ are supposed to be locally strictly elliptic and $b_i \in L^2_{loc}(\mathbb{R}^d,\mu)$, $1 \le i \le d$. We do not assume that L is symmetric or that the first order part is a small perturbation of the second order part in any classical sense. Results obtained by V.I. Bogachev, N. Krylov and M. Röckner (cf. [BKR1], [BKR2] and [BR]) show that it is natural to suppose that μ is absolutely continuous w.r.t. the Lebesgue measure dx and that the density admits a representation φ^2, $\varphi \in H^{1,2}_{loc}(U)$. Moreover, we will assume that a_{ij}, $1 \le i,j \le d$, is contained locally in the weighted Sobolev space $H_0^{1,2}(U,\mu)$ (= the closure of $C_0^\infty(U)$ in $L^2(U,\mu)$ w.r.t. the norm given by $\int |\nabla u|^2 + u^2\,d\mu$).

We are mainly concerned with the following three problems:

(i) Construct and analyse extensions $\overline{L}$ of L generating C_0–semigroups $(\overline{T}_t)_{t\ge0}$ on $L^1(U,\mu)$ that are *sub–Markovian* (i.e., $0 \le f \le 1$ implies $0 \le \overline{T}_t f \le 1$). Henceforth, any extension of L generating a C_0–semigroup will be called *maximal*.

(ii) Find conditions on $(a_{ij})_{1\le i,j\le d}$, $(b_i)_{1\le i\le d}$ and μ so that there is only one maximal extension.

(iii) Construct diffusion processes with transition probabilities given by $(\overline{T}_t)_{t\ge0}$.

Concerning problem (i) we will construct in Theorem 1.5 below a maximal extension $\overline{L}$ generating a sub–Markovian C_0–semigroup $(\overline{T}_t)_{t\ge0}$ in such a way that the space $D(\overline{L})_b$ of all bounded functions in the domain $D(\overline{L})$ is contained locally in the weighted Sobolev space $H_0^{1,2}(U,\mu)$. This implies that, although $\overline{L}$ is not symmetric, this maximal extension is still associated with a bilinear form. More precisely, the following representation holds:

$$\sum_{i,j=1}^{d}\int a_{ij}\partial_i u\,\partial_j v\,d\mu + \sum_{i=1}^{d}\int (b_i^0 - b_i)\partial_i u\,v\,d\mu = -\int \overline{L}u\,v\,d\mu \tag{0.3}$$

for all $u \in D(\overline{L})_b$ and $v \in H_0^{1,2}(U,\mu)_0$ (= the space of all elements $v \in H_0^{1,2}(U,\mu)$ with compact support contained in U). Here

$$b_i^0 = \sum_{j=1}^{d}(\partial_j a_{ij} + 2a_{ij}\partial_j\varphi/\varphi)\,, 1 \le i \le d\,, \tag{0.4}$$