On the Power Method in Max Algebra

Ludwig Elsner
Fakultät für Mathematik
Universität Bielefeld
Postfach 100 131
D-33501 Bielefeld, GERMANY

P. van den Driessche*
Department of Mathematics and Statistics
University of Victoria
Victoria, B.C.
V8W 3P4, CANADA

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Abstract

The eigenvalue problem for an irreducible nonnegative matrix $A = [a_{ij}]$ in the max algebra system is $A \otimes x = \lambda x$, where $(A \otimes x)_i = \max\limits_j (a_{ij} x_j)$ and λ turns out to be the maximum circuit geometric mean, $\mu(A)$. A power method algorithm is given to compute $\mu(A)$ and eigenvector x. This method generalizes and simplifies an algorithm due to Braker and Olsder. The algorithm is developed by using results on the convergence of max powers of A. These results are proved using nonnegative matrix theory. Some asymptotic formulas relating $\mu(A)$, the spectral radius and norms are also given.

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pvdd@smart.math.uvic.ca

1 Introduction

For an $n \times n$ matrix A, the conventional eigenequation for eigenvalue λ and corresponding eigenvector x is $Ax = \lambda x$. In the max algebra system, for $A = [a_{ij}]$ nonnegative, the eigenequation is $\max\limits_j(a_{ij}x_j) = \lambda x_i$, for $i = 1, 2, \ldots, n$. Our aim is to give a power method algorithm to compute the eigenvalue and corresponding eigenvectors in the max algebra.

To motivate this aim and to place it in context, we need some definitions and notation. As in [2], [14, IV3] the *max algebra* system consists of the set of nonnegative numbers with sum $a \oplus b = \max\{a, b\}$ and the standard product ab for $a, b \geq 0$. Thus, for compatible nonnegative matrices A, B, their product is denoted by $A \otimes B$ where $(A \otimes B)_{ik} = \max\limits_j(a_{ij}b_{jk})$. In particular, for a vector $x \geq 0$, the product $(A \otimes x)_i = \max\limits_j(a_{ij}x_j)$. The usual associative and distributive laws hold in this algebra.

The *max-plus algebra* system consists of the set $\mathbf{R} \cup \{-\infty\}$ with operations of maximization and addition; see, e.g. [1, 3, 6, 9], and, in particular, [4] for a power algorithm. The max-plus algebra is isomorphic by the exponential map to the max algebra. As illustrated in [2], the max algebra enables us to use powerful tools of nonnegative matrices, monotonicity and analysis. The isomorphism can then be used to give results in the max-plus algebra where the eigenequation arises in the study of discrete event dynamical systems [1].

Throughout Sections 1-3 we assume that $A = [a_{ij}]$ is an $n \times n$ nonnegative matrix. The weighted directed graph $\mathcal{G}(A)$ associated with A has vertex set $\{1, 2, \ldots, n\}$ and an edge (i, j) from i to j with weight a_{ij} if and only if $a_{ij} > 0$. A *circuit* of length k is a sequence of k edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k), (i_k, i_1)$ where $i_1, i_2, \ldots, i_k$ are distinct. This has *circuit product* $a_{i_1,i_2}a_{i_2,i_3} \cdots a_{i_k,i_1}$, with the positive k-th root as the *circuit geometric mean*. The maximum circuit geometric mean in $\mathcal{G}(A)$ is denoted by $\mu(A)$ and plays an important role. A circuit with circuit product equal to $\mu(A)$ is called a *critical circuit*, and vertices on critical circuits are called *critical vertices*. The *critical matrix* of A, denoted by $A^{\mathcal{C}} = [a_{ij}^{\mathcal{C}}]$, is formed from the principal submatrix of A on the rows and columns corresponding to critical vertices by setting $a_{ij}^{\mathcal{C}} = a_{ij}$ if (i, j) is in a critical circuit, and $a_{ij}^{\mathcal{C}} = 0$ otherwise. Thus the critical graph $\mathcal{G}(A^{\mathcal{C}})$ has vertex set $V^{\mathcal{C}} = \{\text{critical vertices}\}$.

The following basic spectral results are well known in the max-plus algebra (see, e.g. [1, 3, 6, 9, 14]), and are proved in [2] for the max algebra.