TEMPERED SUBGROUPS AND REPRESENTATIONS WITH MINIMAL DECAY OF MATRIX COEFFICIENTS

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ABSTRACT. We present a function F for each simple real linear Lie group G with real rank at least 2 such that F bounds from above all the K-matrix coefficients of non-trivial irreducible spherical unitary representations of G where K is a maximal compact subgroup of G. This enables us to determine when a closed subgroup H is a (G, K)- tempered subgroup of G: for example, if the restriction $F|_H$ of F to H lies in $L^{1-\epsilon}(H)$. We also prove that this function F is the best possible for G a real-split group of type A_n or C_n and as a consequence, we obtain that if H is semisimple, then H is a (G, K)- tempered subgroup of G if and only if $F|_H$ lies in $L^1(H)$.

1. INTRODUCTION

Let G be a connected semisimple linear Lie group and K a maximal compact subgroup of G. We say that a unitary representation of G is *spherical* if it has a K-invariant vector. For a unitary spherical representation ρ, we will use the term "K-matrix coefficients of ρ" to refer to its matrix coefficients with respect to K-invariant unit vectors.

In this paper we are interested in the asymptotic behavior of the K-matrix coefficients of spherical unitary representations of G when restricted to a closed subgroup H of G. One motivation comes from the notion "(G, K)-tempered subgroups" of G defined by Margulis [8]. That is, a closed subgroup H of G is called (G, K)-*tempered* if there exists a (positive) function $q \in L^1(H)$ such that for any non-trivial irreducible spherical unitary representation ρ of G, $|\langle\rho(h)v, w\rangle| \leq q(h)\|v\|\|w\|$ for all $h \in H$ and any K- fixed vectors v and w. Note that any compact subgroup of G is a (G, K)- tempered subgroup for a trivial reason. Margulis also showed in [8] that if a closed subgroup H is a (G, K)-tempered subgroup, then for any non-compact subgroup F of H, the quotient G/F does not allow a compact quotient by a discrete subgroup of G (see [5] for a survey on the general problem).

We denote by $\hat{G}$ (resp. $\hat{G}_K$) the set of equivalence classes of non-trivial irreducible unitary (resp. spherical) representations.

In this paper we first present a "good upper bound function" for K-matrix coefficients for all representations in $\hat{G}_K$ for a simple real linear Lie group G with real rank at least 2. Secondly we show that in simple real-split linear Lie group of type A_n or C_n this function is in fact the best possible by exhibiting a spherical representation of G in $\hat{G}_K$ whose

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K-matrix coefficients are bounded below by this function. We now formulate the main results.

The notation $[x]$ denotes the largest integer which is not greater than x.

Theorem A. *Let G be a connected simple real linear Lie group with real rank $n \geq 2$, K a maximal compact subgroup, B a minimal parabolic subgroup, $A \subset B$ a maximal $\mathbb{R}$-split torus, $A^+ \subset A$ the positive Weyl chamber given by the choice of B. Denote by Φ' the set of all non-multipliable roots in the relative root system $\Phi_{\mathbb{R}}(A, G)$ and by $\alpha_1, ..., \alpha_n$ the basis of Φ' whose subscripts are determined by the highest weight given in section 2.1.*

Then for any $\epsilon > 0$, there exists a constant C (depending on ϵ) such that for any $\rho \in \hat{G}_K$ and f_0 a K-invariant unit vector of ρ, the K-matrix coefficient $\langle \rho(g) f_0, f_0 \rangle$ satisfies

$$|\langle \rho(g) f_0, f_0 \rangle| \leqq C \cdot F(g)^{1-\epsilon} \qquad \text{for any } g \in G$$

where F is the K-bi-invariant function defined on A^+ as follows according to the type of Φ':

Φ'	F
$A_n, n \geq 2$	$\begin{cases} (\prod_{i=1}^{n-1/2} \alpha_i{}^{-i/2})(\prod_{i=n+1/2}^{n} \alpha_i{}^{-(n-i+1)/2}) & \text{for } n \text{ odd} \\ (\prod_{i=1}^{n/2} \alpha_i{}^{-i/2})\alpha_{n/2+1}{}^{-n/4}(\prod_{i=n/2+2}^{n} \alpha_i{}^{-(n-i+1)/2}) & \text{for } n \text{ even} \end{cases}$
$B_n, n \geq 2$	$\left(\prod_{i=1}^{[n/2]} \alpha_i{}^{-i}\right)\left(\prod_{i=[n/2+1]}^{n} \alpha_i{}^{-n/2}\right)$
$C_n, n \geq 2$	$\left(\prod_{i=1}^{n-1} \alpha_i{}^{-i}\right)\alpha_n{}^{-n/2}$
$D_n, n \geq 4$	$\left(\prod_{i=1}^{[n/2]} \alpha_i{}^{-i}\right)\left(\prod_{i=[n/2+1]}^{n-2} \alpha_i{}^{-n/2}\right)(\alpha_{n-1}\alpha_n)^{-n/4}$
E_6	$(\alpha_1\alpha_2{}^2\alpha_3^2\alpha_4^3\alpha_5^2\alpha_6)^{-1}$
E_7	$(\alpha_1{}^2\alpha_2^{7/2}\alpha_3^4\alpha_4^6\alpha_5^{9/2}\alpha_6^3\alpha_7^{3/2})^{-1}$
E_8	$(\alpha_1^2\alpha_2^4\alpha_3^5\alpha_4^8\alpha_5^7\alpha_6^5\alpha_7^3\alpha_8)^{-1}$
F_4	$(\alpha_1^2\alpha_2^3\alpha_3^4\alpha_4^2)^{-1}$
G_2	$(\alpha_1^2\alpha_2)^{-1}$

Corollary B. *Let G, K, B, A, A^+ and F be as in Theorem A and let H be a closed subgroup of G. If the restriction $F|_H$ of F to H is in $L^{1-\epsilon}(H)$ for some $\epsilon > 0$, then H is a (G, K)-tempered subgroup of G.*