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LOW-DIMENSIONAL LINEAR REPRESENTATIONS OF $\mathrm{Aut}\, F_n,\ n \geq 3$

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Abstract. We classify all complex representations of Aut F_n, the automorphism group of the free group F_n ($n \geq 3$), of dimension $\leq 2n-2$. Among those representations is a new representation of dimension $n+1$ which doesn't vanish on the group of inner automorphisms.

Introduction

This paper continues the study of low-dimensional linear representations of $\Gamma_n = \mathrm{Aut}\, F_n$ ($n \geq 3$), the automorphism group of the free group, begun in [R1] (low-dimensional representations of Aut F_2 were analyzed in [DP]). It was shown in [R1] (cf. also Theorem 1.2 below) that any n-dimensional representation of Aut F_n factors through the the canonical homomorphism $f\colon \Gamma_n \to GL_n(\mathbf{Z})$. In this paper we will show that in dimension higher than n, the group Γ_n acquires new representations. Namely, we will establish the existence of a homomorphism $g\colon \Gamma_n \to GL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n$ which "lifts" f and gives rise to an $(n+1)$-dimensional representation. The main result of the paper (Theorem 3.1) claims that this representation accounts in fact for all representations of Γ_n of dimension $\leq 2n-2$ (in particular, any such representation factors through g). We also use the homomorphism g to construct an infinite family of pairwise inequivalent representations of Γ_n (Proposition 2.4).

1. Notations and preliminaries.

Throughout the paper, E_m will denote the identity $m \times m$-matrix; for a partition $m = m_1 + \cdots + m_k$, $G(m_1, \ldots, m_k)$ will denote the product

$$GL_{m_1}(\mathbf{C}) \times \cdots \times GL_{m_k}(\mathbf{C})$$

diagonally embedded in $GL_m(\mathbf{C})$, with $p_i(m_1, \ldots, m_k)\colon G(m_1, \ldots, m_k) \to GL_{m_i}(\mathbf{C})$ being the corresponding projection; S_n is the symmetric group on $\{1, \ldots, n\}$. To comply with the classical tradition, Γ_n and S_n will be treated as groups of *right* transformations (in other words, the result of application of f to a will be written as $(a)f$, with the law of composition given by: $(a)(fg) = ((a)f)g$).

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We will use the following elements of $\Gamma_n = \mathrm{Aut}\, F_n$:

$$\rho_{ij}\colon \begin{cases} x_i & \to & x_i x_j, \\ x_k & \to & x_k, \quad k \neq i \end{cases} \qquad \lambda_{ij}\colon \begin{cases} x_i & \to & x_j x_i, \\ x_k & \to & x_k, \quad k \neq i \end{cases}$$

$$\varepsilon_i\colon \begin{cases} x_i & \to & x_i^{-1}, \\ x_k & \to & x_k, \quad k \neq i \end{cases} \qquad \begin{array}{l} \varepsilon_{ij} = \varepsilon_i \varepsilon_j; \\ \varepsilon = \varepsilon_1 \dots \varepsilon_n \colon x_i \to x_i^{-1} \quad \forall i, \end{array}$$

$$\forall \pi \in S_n, \quad \sigma_\pi \colon x_i \to x_{(i)\pi}.$$

It is known that these elements generate Γ_n (cf. [MKS]). One easily checks the following identities:

$$\varepsilon_i^2 = \mathrm{id}_{F_n}, \qquad \varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i, \qquad \sigma_\pi^{-1} \varepsilon_i \sigma_\pi = \varepsilon_{(i)\pi}$$

(1)

$$\varepsilon_i^{-1} \rho_{ij} \varepsilon_i = \lambda_{ij}^{-1}, \qquad \varepsilon_{ij}^{-1} \rho_{ij} \varepsilon_{ij} = \varepsilon^{-1} \rho_{ij} \varepsilon = \lambda_{ij} \;\; \forall i, j, \;\; i \neq j.$$

It follows that H_n, the subgroup generated by all ε_i, is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n$; all σ_π, $\pi \in S_n$, form a subgroup Σ_n which is a replica of S_n; and the subgroup Ω_n, generated by all ε_i and σ_π together is the semi-direct product $\Sigma_n \ltimes H_n$.

A special feature that distinguishes the case $n \geq 3$ from the case $n = 2$ is the existence of the following commutator relations:

$$[\lambda_{ij}, \lambda_{jk}] = \lambda_{ik} \quad , \quad [\rho_{ij}, \rho_{jk}] = \rho_{ik} \tag{2}$$

(i, j, and k are all distinct) where $[x, y] = xyx^{-1}y^{-1}$.

Passing to abelianization, we obtain a (surjective) homomorphism

$$f\colon \Gamma_n \to \mathrm{Aut}(F_n/[F_n, F_n]) = GL_n(\mathbf{Z})$$

(matrix presentation is taken with respect to the basis of $F_n/[F_n, F_n] \simeq \mathbf{Z}^n$ made up of the images of $x_1, \dots, x_n$; for consistency, $\mathbf{Z}^n$ is treated as *n-rows* of integers, with *right* action of $GL_n(\mathbf{Z})$). Let $N = \mathrm{Ker}\, f$. Our argument makes essential use of the fact (cf. [LS], p.28) that N as a normal subgroup of Γ_n is generated by the element

$$\beta = \lambda_{n,n-1}^{-1} \rho_{n,n-1}.$$

Another element to be frequently used is

$$\gamma = \lambda_{n,n-1} \rho_{n,n-1}.$$

Lemma 1.1. *Let $\varphi\colon \Gamma_n \to G$ be a homomorphism of Γ_n to some group G. If the restriction of φ to Ω_n is not injective, then φ factors through f.*

Proof. First, assume that the restriction $\varphi \mid H_n$ is not injective. It is easy to see that H_n has only two proper subgroups normalized by Σ_n : $\langle \varepsilon \rangle$ and $H' = \{h \in H_n \mid \det f(h) = 1\}$. If $\mathrm{Ker}\, f = \langle \varepsilon \rangle$, then (cf. (1))

$$\varphi(\lambda_{ij}) = \varphi(\varepsilon \rho_{ij} \varepsilon^{-1}) = \varphi(\rho_{ij})$$