Geometrical Bijections in Discrete Lattices

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1 Introduction

When solving Tarski's circle squaring problem Laczkovich introduced the notion of a "uniformly spread set". Taking a closer look at the combinatorial ideas behind his approach we consider mappings with a "wobbling property". A mapping $f : X \to Y$ (X, Y subsets of a metric space) is a wobbling mapping if no point in X is moved too far, formally if

$$\sup_{x \in X} d\,(x, f(x)) \; < \; \infty \; .$$

Wobbling mappings occur in many real world situations. Rounding in numerical analysis, image processing, distortion of crystals, but also earthquakes are typical.
We call subsets of a metric space "equivalent" if there exists a wobbling bijection between them. In particular uniformly spraed sets are those which a equivalent to a lattice $\mathbb{Z}^n$ considered as Euclidean space.
The paper is organized as follows: After a short technical introduction giving general useful information also on continued fractions which will play a crucial role in the effective construction of certain wobbling bijections.
In a chapter on geometrical constructions we shall give a constructive proof for the fact that any linear mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ with determinant 1 transforms the lattice $\mathbb{Z}^n$ into a uniformly spread set. It

suffices to consider the Jordan normal form with the typical matrices

- $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ shearing
- $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ rotation
- $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ rectangular lattice

In fact shearings will play an important role for an engineering approach to such problems leading to computer visualization. In particular this method already yields a uniform upper bound in terms of a matrix norm (theorem 4.1) for the wobbling distances, which was not obvious from the beginning.

Also by means of continued fractions and the Sturmian sequence from billiards theory we will indicate another efficient constructive method and explain this in particular for rectangular lattices. As a side result we shall investigate the discrepancy of a billards problem. We shall conclude the paper with results on recursive wobblings and an outlook on paradoxical situations [1].

2 General definitions

By $\mathbb{N} = \{1, 2, 3, \dots\}$ we denote the natural numbers. $\mathbb{N}_0$ stands for the set of nonnegative integers and $\mathbb{R}^+$ for the set of nonnegative reals. For a rational or real $x \in \mathbb{R}$ let $[x]$ be the nearest integer, where ties are broken by rounding to the smaller integer. Let (M, d) be a metric space and $X, Y \subseteq M$. A mapping $f : X \to Y$ is a wobbling mapping if $\sup_{x\in X} d(x,\, f(x)) < \infty$ and the supremun is called the wobbling distance of f. The composition of two wobbling mappings has the same quality and this also holds for the inverse of a wobbling bijection. X and Y are called "wobbling equivalent" or "equivalent" ($X \equiv Y$) if there exists a wobbling bijection from X onto Y. The classical Cantor-Bernstein argument shows that X and Y are equivalent iff there exist wobbling injections $f : X \to Y$ and $g : Y \to X$. As we shall consider lattices in the Euclidean plane with the metric given by the Euclidean norm $\|.\|$ it is noteworthy that for countable discrete metric spaces one can use a graph theoretic approach to wobbling mappings using the Hall-Rado theorem:
For $k \in \mathbb{R}^+$ and $X, Y \subseteq M$ let $C_k(X, Y)$ be the bipartite $k-$neighborhood graph with parts X, Y and edges defined by $d(x, y) \leq k$. Let X, Y be subsets of a countable discrete metric space. Then the following statements are equivalent:

(i) There exists an injective wobbling mapping $f : X \to Y$.

(ii) There exists k, such that $C_k(X, Y)$ satisfies the Hall condition: For every finite subset X' of X the cardinality $|N_k(x) \bigcap Y|$ of the $k-$neighborhood of X' in Y satifies $|N_k(x) \bigcap Y| \geq |X'|$.

The continued fraction expansion $[a_0, a_1, ...]$, $a_i \in 0, 1, 2...$ of a real $1/\alpha > 0$ is defined by $1/\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$. Conversely it is easy to exhibit the continued fraction expansion of any $1/\alpha$ by the generalized eulidean algorithm for the pair $(1, \alpha)$. Let

$$\begin{aligned} 1 &= \alpha a_0 + r_0 \\ \alpha &= r_0 a_1 + r_1 \\ r_0 &= r_1 a_2 + r_2 \\ r_i &= r_{i+1} a_{i+2} + r_{i+2} \text{ for } i \in \mathbb{N}. \end{aligned} \tag{2.1}$$