REALIZATIONS OF FROBENIUS FUNCTIONS

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Abstract. In an earlier paper (see [ALR]) we studied the so-called Frobenius functions on certain translation quivers. Here we show that the classification given there is in some sense complete: every Frobenius length function on the wing $W(n)$ and the tube $T(n)$ is equivalent to the length function on a convex subquiver of the Auslander–Reiten quiver of the module category over some algebra A.

1. Introduction

Let $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ be a translation quiver (without multiple arrows), and let $f : \Gamma_0 \longrightarrow \mathbb{Z}$ be an integral valued function defined on the vertices of Γ. For any $z \in \Gamma_0$ a non-projective vertex we define the defect of the function f at the vertex z (or rather on the mesh ending at z) by $\delta(z) = \delta_f(z) = f(z) + f(\tau z) - \sum_{y \to z} f(y)$. We call the function f a *Frobenius function* if $\delta(z) \neq 0$ implies $\delta(z) > f(z)$ and $\delta(z) > f(\tau z)$. A Frobenius function with positive values only will said to be *positive*. Meshes with non-zero defect are called *incomplete meshes*; otherwise we say that f is additive on the mesh. Two Frobenius functions are said to be *equivalent* provided they have the same set of incomplete meshes. A typical example of a positive Frobenius function is the dimension function on the stable Auslander–Reiten quiver of a finite dimensional selfinjective algebra over a field k. A *Frobenius length function* is a positive Frobenius function for which $\delta(z) - f(z) = \delta(z) - f(\tau z) = 1$ whenever the mesh ending at the vertex z is incomplete. A typical example of a Frobenius (length) function is the length function ℓ on the stable Auslander–Reiten quiver of a finite dimensional selfinjective algebra over a field k.

In [ALR] we studied in detail positive Frobenius functions on certain translation quivers. In particular we investigated the cases of the wings $W(n)$, the stable tubes $T(n)$ and the related translation quivers $\mathbb{Z}A_\infty$ and $\mathbb{Z}A_\infty^\infty$ and gave a full description of the equivalence classes of Frobenius functions on these quivers by describing geometrically the possible configurations of incomplete meshes.

The aim of this paper is to give algebraic realizations for Frobenius length functions on $W(n)$ and $T(n)$. Namely, we want to prove the following:

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THEOREM 1. *Let f be a positive basic Frobenius length function on the translation quiver $W(n)$. Then there exists a finite dimensional, basic special biserial algebra $\mathcal{C}$ and a translation subquiver Γ of the Auslander–Reiten quiver $\Gamma(\mathcal{C})$ which is isomorphic to (hence it can be identified with) $W(n)$ so that:*

(i) *the length function on Γ coincides with f;*

(ii) *the incomplete meshes of Γ with respect to f are precisely those from which a projective-injective vertex in the Auslander–Reiten quiver $\Gamma(\mathcal{C})$ was removed to obtain Γ.*

We note here that every Frobenius length function on $W(n)$ is equivalent to a basic Frobenius length function (see [ALR]). Thus the above theorem gives algebraic realization up to equivalence for an arbitrary Frobenius length function on $W(n)$.

THEOREM 2. *Let f be a positive Frobenius length function on the translation quiver $T(n)$. Then there exists a finite dimensional special biserial algebra $\mathcal{C}$ and a translation subquiver Γ of the Auslander–Reiten quiver $\Gamma(\mathcal{C})$ which is isomorphic to (hence it can be identified with) $T(n)$ so that:*

(i) *the length function on Γ is equivalent to f;*

(ii) *Γ can be obtained from a component of $\Gamma(\mathcal{C})$ by removing all the projective-injective vertices;*

(iii) *the incomplete meshes of Γ with respect to f are precisely those from which a projective-injective vertex in the Auslander–Reiten quiver $\Gamma(\mathcal{C})$ was removed.*

We would like to mention here that the ideas used in our construction very much rely on ideas contained in the works [HW] and [WW]. In particular we use the explicit description of the Auslander–Reiten sequences for special biserial algebras, given in [WW] (see also [BR]). For the unexplained notation and background we refer to [ALR] and [R].

2. Construction of the algebras

In what follows, k will be an arbitrary field. We shall consider k-algebras (not necessarily finite dimensional) which will be given as path algebras modulo some relations over finite quivers.

Thus let $\mathcal{C}$ be such a k-algebra. The considered $\mathcal{C}$-modules will always be left modules. Accordingly, we shall 'multiply the arrows' of the quiver from the left: if α and β are arrows in the quiver then $\alpha\beta$ will stand for the path with β coming first, followed by α. The category of all left $\mathcal{C}$-modules which are finite dimensional over k will be denoted by $\mathcal{C}$-mod.

Throughout the paper we shall use the following coordinatization for the translation quivers under consideration. The set of vertices of $\mathbb{Z}A_\infty^\infty$ is the the set $\mathbb{Z} \times \mathbb{Z}$ of integral lattice points in the plane. There are arrows $(i,j) \to (i+1,j)$ and $(i,j) \to (i,j+1)$, and the translation is defined by $\tau(i,j) = (i-1,j-1)$ for any $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. We consider $\mathbb{Z}A_\infty$ as the full subquiver of $\mathbb{Z}A_\infty^\infty$ on the set of vertices $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} \,|\, i \leq j\}$. For $W(n)$ we fix a standard embedding, mapping the projective vertices $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ to the points $(1,1), (1,2), \ldots, (1,n)$. Hence, $W(n) = \{(i,j) \,|\, 1 \leq i \leq j \leq n\}$. Finally, we also consider $\mathbb{Z}A_\infty$ as the universal cover of $T(n)$ so that the vertices on the mouth of $T(n)$ correspond to the boundary $\{(i,i) \,|\, i \in \mathbb{Z}\}$ of $\mathbb{Z}A_\infty$. The standard coordinatization of $T(n)$ maps the points of $T(n)$ to the strip $\{(i,j) \,|\, 1 \leq i \leq j,\ i \leq n\}$.