Chern Classes of Tautological Sheaves on Hilbert Schemes of Points on Surfaces

Manfred Lehn

Introduction

Hilbert schemes $X^{[n]}$ of n-tuples of points on a complex projective manifold X are natural compactifications of the configuration spaces of unordered distinct n-tuples of points on X. Their geometry is determined by the geometry of X itself and the geometry of the 'punctual' Hilbert schemes of all zero-dimensional subschemes in affine space that are supported at the origin. Thus one is naturally led to the following problem:

Determine explicitly the geometric or topological invariants of the Hilbert schemes $X^{[n]}$ such as the Betti numbers, the Hodge numbers, the Chern numbers, the cohomology ring, from the corresponding data of the manifold X itself.

This problem is most attractive when X is a surface, since then the Hilbert schemes are themselves irreducible projective manifolds, by a result of Fogarty [9], whereas for higher dimensional varieties the Hilbert schemes are in general neither irreducible nor smooth nor pure of expected dimension.

In the surface case, the answer to the problem above for the Betti numbers was first given by Göttsche in [11]. The answer turns out to be particularly beautiful (cf. Theorem 2.1 below). The problem for the Hodge numbers was solved by Sörgel and Göttsche [12]. For a different approach to both results see [3]. The answer for the Chern classes will be implicitly given in a forthcoming paper by Ellingsrud, Göttsche and the author [4].

The question for the ring structure of the cohomology is more difficult. In general, $X^{[2]}$ is the quotient of the blow-up of $X \times X$ along the diagonal by the canonical involution that exchanges the factors. Thus the case of interest is $H^*(X^{[n]})$, $n \geq 3$. The ring structure was found for $(\mathbb{P}^2)^{[3]}$ by Ellingsrud and Strømme [5], and for $X^{[3]}$, X smooth projective of arbitrary dimension, by Fantechi and Göttsche [8]. In another direction, Ellingsrud and Strømme [6] gave generators for $H^*((\mathbb{P}^2)^{[n]}, \mathbb{Z})$, n arbitrary, and an implicit description of the relations.

Vafa and Witten [27] remarked that Göttsche's Formula for the Betti numbers is identical with the Poincaré series of a Fock space modelled on the cohomology of X. Nakajima [21] succeeded in giving a geometric construction of such a Fock space structure on the cohomology of the Hilbert schemes, leading to a natural 'explanation' of Göttsche's result. Similar results have been announced by Grojnowski [13].

Following the presentation of Grojnowski, this can be made more precise as follows: sending a pair (ξ', ξ'') of subschemes of length n' and n'', respectively, and of disjoint support to their union $\xi' \cup \xi''$ defines a rational map

$$m : X^{[n']} \times X^{[n'']} - - \to X^{[n'+n'']}.$$

This map induces linear maps on the rational cohomology

$$m_{n',n''} : H^*(X^{[n']}; \mathbb{Q}) \otimes H^*(X^{[n'']}; \mathbb{Q}) \longrightarrow H^*(X^{[n'+n'']}; \mathbb{Q})$$

and

$$m^{n',n''} : H^*(X^{[n'+n'']}; \mathbb{Q}) \longrightarrow H^*(X^{[n']}; \mathbb{Q}) \otimes H^*(X^{[n'']}; \mathbb{Q}).$$

If we let $\mathbb{H} := \oplus_n H^*(X^{[n]}; \mathbb{Q})$, then these maps define a multiplication and a comultiplication

$$m_* : \mathbb{H} \otimes \mathbb{H} \longrightarrow \mathbb{H}, \qquad m^* : \mathbb{H} \longrightarrow \mathbb{H} \otimes \mathbb{H},$$

which make $\mathbb{H}$ a commutative and cocommutative bigraded Hopf algebra. The result of Nakajima and Grojnowski says that this Hopf algebra is isomorphic to the graded symmetric algebra of the vector space $H^*(X; \mathbb{Q}) \otimes t\mathbb{Q}[t]$.

More explicitly, Nakajima constructed linear maps [1]

$$\mathfrak{q}_n : H^*(X; \mathbb{Q}) \longrightarrow \mathrm{End}_{\mathbb{Q}}(\mathbb{H}), \quad n \in \mathbb{Z},$$

and proved that they satisfy the 'oscillator' or 'Heisenberg' relations

$$[\mathfrak{q}_n(\alpha), \mathfrak{q}_m(\beta)] = (-1)^n \cdot n \cdot \delta_{n+m} \cdot \int_X \alpha\beta \cdot \mathrm{id}_{\mathbb{H}}.$$

Here the commutator is to be taken in a graded sense.

The multiplication and the comultiplication of $\mathbb{H}$ are not obviously related to the quite different ring structure of $\mathbb{H}$, which is given by the usual cup product on each direct summand $H^*(X^{[n]}; \mathbb{Q})$. (Strictly speaking, $\mathbb{H}$ contains a countable number of idempotents $1_{X^{[n]}} \in H^0(X^{[n]}; \mathbb{Q})$ but not a unit unless we pass to some completion).

This paper attempts to relate the Hopf algebra structure and the cup product structure. More precisely:

Let F be locally free sheaf of rank r on X. Attaching to a point $\xi \in X^{[n]}$, i.e. a zero-dimensional subscheme $\xi \subset X$, the $\mathbb{C}$-vector space $F \otimes \mathcal{O}_\xi$ defines a locally free sheaf $F^{[n]}$ of rank rn on $X^{[n]}$. The Chern classes of all sheaves on $X^{[n]}$ of this type generate a subalgebra $\mathcal{A} \subset \mathbb{H}$. We will describe a purely algebraic algorithm to determine the action of $\mathcal{A}$ on $\mathbb{H}$ in terms of the $\mathbb{Q}$-basis of $\mathbb{H}$ provided by Nakajima's results. We collect the Chern classes of all sheaves $F^{[n]}$ for a given sheaf F into operators

$$\mathfrak{ch}(F) : \mathbb{H} \to \mathbb{H}, \qquad \mathfrak{c}(F) : \mathbb{H} \to \mathbb{H}$$

and geometrically compute the commutators of these operators with the 'standard operators' defined by Nakajima.

[1] Our presentation differs in notations and conventions slightly from Nakajima's.