On some Poincaré–series on hyperbolic space

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1 Introduction

Let $\mathbb{H}^n$ denote hyperbolic n-space with $d(P,Q)$ the hyperbolic distance between the points P and Q. Introduce

$$\delta(P,Q) = 2\cosh(d(P,Q))$$

then it is easily seen (see e.g. [5]) that

$$\delta(P,Q) = \frac{\sum_{i=0}^{n-2}\left(u_i(P)-u_i(Q)\right)^2 + v^2(P) + v^2(Q)}{v(P)v(Q)}$$

where $u_0, \ldots, u_{n-2}, v$ are the coordinates in the upper half-space model, see section 2. From the definition follows that δ is a point pair invariant, i.e. for γ from the isometry group of $\mathbb{H}^n$ we have

$$\delta(\gamma P, Q) = \delta(P, \gamma^{-1}Q).$$

Let Γ be a discrete cofinite subgroup of $\mathrm{Iso}^+(\mathbb{H}^n)$, of orientation preserving isometries of $\mathbb{H}^n$. Then the Poincaré series built from δ with respect to Γ is defined as

$$H(P,Q,s) := \sum_{\gamma\in\Gamma} \left(\delta(P,\gamma Q)\right)^{-(n-2+s)}$$

for $\Re s$ sufficiently large. Analogous considerations as in [4], section 3.1 show that this Poincaré series converges absolutely for all P, $Q \in \mathbb{H}^n$ if $\Re s > 1$.

$H(P,Q,s)$ plays an important role for the study of the distribution of hyperbolic lattice points. Its analytic behaviour was investigated by Huber [6] for $n=2$ and for $n=3$ an extensive study can be found in the monograph of Elstrodt, Grunewald and Mennicke, see [4].

In the present paper we relate $H(P,Q,s)$ (for any dimension n) with certain Theta lifts of real analytic Poincaré series of Selberg type defined on $\mathbb{H}^2$. We have already considered Theta lifts of this sort in [13] and [12].

For more information about such Theta lifts generalizing Shimura's lift [16] see also the work of Niwa [14], Shintani [17], Duke [3] and Katok/Sarnak, [7].

Kudla and Millson, see [8], do similar computations in the setting of holomorphic Hilbert–Siegel modular forms but restrict to the case, that the manifold for the lifted forms is compact and is at most three–dimensional .

In the present paper we have the following situation: Let Q denote a quadratic form of signature $(1,n)$. Then for $z \in \mathbb{H}^n$, $w \in \mathbb{H}^n$ Siegel's theta function $\Theta_{Q,\mathbb{Z}^{n+1}}(z,w)$ will be defined in section 6. For ν a fundamental discriminant the real analytic Poincaré series for $\Gamma_0(N)$ of weight $(1-n)/2$ and multiplier $v_{D,n}$ is given as

$$P_\nu(z,s) = \sum_{T=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(N)} \overline{v_{D,n}(T) e^{\frac{i(n-1)}{2} \arg(cz+d)}} y_T^s e(\nu x_T + i|\nu| y_T), \quad \Re s > 1,$$

with $x_T + iy_T := Tz$. See sections 6,7 for the full definitions. The lifted series is defined as $P_\nu^\Theta(w,s) = \int_{\mathcal{F}_N} P_\nu(z,s) y^{\frac{n+1}{4}} \overline{\Theta(z,w)} d\mu(z)$ where $\mathcal{F}_N$ denotes a fundamental domain for $\Gamma_0(N)$.

Our main result now is the following

Theorem 1.1. *Let $\Gamma(Q)$ be the group of units for Q which is a discrete cofinite subgroup of $\mathrm{Iso}^+(\mathbb{H}^n)$, ν a fundamental discriminant integer. Let further $\mathbf{a}_i$, $i \in I$, be a full set of orbit representants for the action of $\Gamma(Q)$ on $\{\mathbf{a} \in \mathbb{Z}^{n+1} : Q[\mathbf{a}] = \nu\}$, see section 7, with isotropy group $\mathcal{P}_i$ which has finite order f_i if $\nu > 0$.*

Let the points η_i and the geodesic hyperplanes $\mathbf{g}_i$ associated to $\mathbf{a}_i$ be given as in the end of section 9, and let $d\sigma$ denote hyperbolic measure on such a hyperplane. We then have for $\Re s > (n+1)/4$

i) if $\nu > 0$

$$P_\nu^\Theta(w,s) = \frac{\Gamma(s+\frac{n-3}{4})}{f_i(4\pi\nu)^{s+\frac{n-3}{4}}} \sum_{i \in I} H(\eta_i, w, 2s - \frac{n-1}{2}), \quad w \in \mathbb{H}^n,$$

ii) if $\nu < 0$

$$\begin{aligned} P_\nu^\Theta(w,s) &= \frac{1}{2^{4s+n-4}\pi^{s+\frac{3n-7}{4}}|\nu|^{s+\frac{n-3}{4}}} \frac{\Gamma(2s+\frac{n-3}{2})}{\Gamma(s-\frac{n-1}{4})} \\ &\quad \times \sum_{i\in I} \int_{\mathcal{P}_i \backslash \mathbf{g}_i} H(\omega, w, 2s - \frac{n-1}{2}) d\sigma(\omega), \quad w \in \mathbb{H}^n. \end{aligned}$$

One should remark, that $H(P,Q,s)$ can be chosen as a kernel function in the Selberg trace formula. For $n = 3$ its spectral expansion has been computed in [4] and for Γ a discrete cofinite subgroup of $\mathrm{PSL}_2(\mathbb{C})$ it has the form

$$\begin{aligned} H(P,Q,s) &= \frac{2^s\pi}{\mathrm{vol}(\Gamma)} \frac{\Gamma(\frac{s-1}{2})\Gamma(\frac{s+1}{2})}{\Gamma(s+1)} \\ &+ 2^s\pi \sum_m \frac{\Gamma(\frac{s-\mu_m}{2})\Gamma(\frac{s+\mu_m}{2})}{\Gamma(s+1)} e_m(Q)\overline{e_m(P)} \qquad (1) \\ &+ \text{ Eisenstein terms} \end{aligned}$$

where the sum is over the nonconstant L^2–eigenfunctions e_m of the Laplace–Beltrami operator with eigenvalue $\lambda_m = 1 - \mu_m^2$ which we assume to be orthonormalized.