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A CLASSIFICATION OF PARABOLIC SUBGROUPS OF CLASSICAL GROUPS WITH A FINITE NUMBER OF ORBITS ON THE UNIPOTENT RADICAL

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Abstract. Let G be a classical algebraic group defined over an algebraically closed field. We classify all instances when a parabolic subgroup P of G acts on its unipotent radical P_u, or on $\mathfrak{p}_u$, the Lie algebra of P_u, with only a finite number of orbits.

The proof proceeds in two parts. First we obtain a reduction to the case of general linear groups. In a second step, a solution for these is achieved by studying the representation theory of a particular quiver with certain relations.

Furthermore, for the general linear groups we obtain a combinatorial formula for the number of orbits in the finite cases.

1. Introduction

Let G be a reductive algebraic group defined over an algebraically closed field k. Let P be a parabolic subgroup of G with unipotent radical P_u. Then P acts on P_u via conjugation and on $\mathfrak{p}_u$, the Lie algebra of P_u, via the adjoint action. The modality of the action of P on $\mathfrak{p}_u$, denoted by $\operatorname{mod}(P : \mathfrak{p}_u)$, is the maximal number of parameters upon which a family of P-orbits on $\mathfrak{p}_u$ depends; likewise for $\operatorname{mod}(P : P_u)$, see Section 2 for a precise definition. The *modality* of P is defined as $\operatorname{mod} P := \operatorname{mod}(P : \mathfrak{p}_u)$. Note that $\operatorname{mod} P$ is zero precisely when P has only a finite number of orbits on $\mathfrak{p}_u$. By a fundamental result of R.W. Richardson P always admits a dense orbit on P_u [Ri]. Thus, if $\operatorname{mod} P$ is positive, an infinite family of orbits occurs in a subvariety of P_u of codimension at least 1. The general problem of classifying all parabolic subgroups of reductive algebraic groups G of modality zero was posed in [PR]. In this paper we present a classification for the classical groups. Let

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$\ell(P_u)$ denote the nilpotency class of P_u. Our chief result is

Theorem 1.1. *Let G be a simple classical algebraic group and P a parabolic subgroup of G. Suppose that* $\operatorname{char} k$ *is either zero or a good prime for G. The number of P-orbits on P_u, or $\mathfrak{p}_u$, is finite if and only if one of the following holds:*

(i) $\ell(P_u) \leq 4$;
(ii) *G is of type D_r, $\ell(P_u) = 5$, P is not invariant under the automorphism of G of order 2 stemming from the interchange of the simple roots α_{r-1} and α_r, and the derived subgroup of a Levi subgroup of P consists of two commuting simple parts.*

Under the assumption that $\operatorname{char} k = 0$ a partial classification of parabolic subgroups as in Theorem 1.1 was obtained in [HR1]. The cases for Borel subgroups, semisimple rank one, and semisimple rank two parabolic subgroups of all simple algebraic groups are classified in [Ka], [PR], and [JR], respectively. A detailed discussion of parabolic subgroups with an Abelian unipotent radical, including the structure of orbit closures, appears in [RRS]. Further results on parabolic subgroups with this finiteness property in exceptional groups can also be found in [JR].

Our proof of Theorem 1.1 proceeds in two steps. The first one consists in a reduction of the problem to general linear groups, and the second in a solution for this latter case. However, our result for the general linear groups is more general than the finiteness statement of Theorem 1.1, as in this instance we can relax the requirement on the underlying field k.

Theorem 1.2. *Let k be an infinite field and let V be a finite-dimensional k-vector space. Let P be a parabolic subgroup of $GL(V)$. Then the number of P-orbits on P_u or $\mathfrak{p}_u$ is finite if and only if $\ell(P_u) \leq 4$.*

Theorem 1.2 is proved in Section 4. This is accomplished by translating the problem into a question of the representation-type of a particular category of representations of a quiver with certain relations. In turn, this can be answered using a result from the theory of Δ-filtered modules over quasi-hereditary algebras from [DR]. Apart from the desired finiteness statement, this approach also leads to a formula for the number of orbits when $\ell(P_u) \leq 4$.

Theorem 1.2 is generalised in a forthcoming paper by T. Brüstle and the first author [BrHi], where all parabolic subgroups P of $GL(V)$ with a finite number of orbits on $\mathfrak{p}_u^{(l)}$ for $l \geq 1$ are classified, where $\mathfrak{p}_u^{(l)}$ denotes the l-th member of the descending central series of $\mathfrak{p}_u$.

In a related paper by the first author [Hi] the action of a maximal torus $T \subset B$ of $GL(V)$ on subalgebras of $\mathfrak{b}_u$ is investigated; a complete geometric description of the T-orbits is given in terms of the moduli spaces of the action of T.