Defining k in $G(k)$

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We show how the field of definition k, of a k-isotropic absolutely almost simple k-group G "lives" in the group $G(k)$ of k-rational points. The construction which is inspired by the fundamental work of Borel-Tits [BT3] is as follows: We choose an element inside the center of the unipotent radical of a minimal parabolic k-subgroup P; the orbit under the action of the center Z of a Levi k-subgroup of P generates a one-dimensional vector space which then carries the additive field structure in a natural way. The multiplicative structure is induced by the action of Z. If G is k-simple, our construction yields a finite extension l of k.

As an immediate consequence we obtain an answer to a question of Borovik-Nesin [BN] p. 367 under the additional assumption that G is k-isotropic:

Theorem *If G is a k-simple k-isotropic group such that $G(k)$ has finite Morley rank, then k is either algebraically closed or real closed. If G is absolutely simple k-isotropic, then k is algebraically closed.*

1 Preliminaries

The model theoretic notion of defining a field in a group means giving a first order formula in the language of groups (just using the group operation and possibly naming finitely many elements of the group as parameters) satisfied by a set of elements of the group such that the field operations on this set are again definable by first order formulas. In this way the field "lives" inside the group. Our construction yields just such a realization of the field in the group.

We assume that the reader is familiar with the basic theory of linear algebraic groups. Most of what is needed here can be found in [Bo] or [BT1]. Throughout this note, k is a field, $K \supseteq k$ its algebraic closure, and G is a reductive k-group. Even though from the model theoretic point of view the question of definability becomes

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somewhat trivial if k is finite, we still think it worthwhile having a uniform treatment for all fields.

A reductive group is always assumed to be connected. As usual let G_m and G_a denote the multiplicative and the additive group of K, respectively.

A group G is called k-almost simple if it has no proper normal connected subgroup which is defined over k. If G is almost simple, then it is of course k-almost simple; in this case, G is also called absolutely almost simple. The converse is false, as a k-simple group need not be absolutely simple; see the comments in Section 4 below.

Let T be a maximal torus of G defined over k. Then T decomposes into an almost direct product $T = T_a \cdot T_d$, where T_d is k-split and T_a is anisotropic.

A Borel subgroup of G is a maximal connected solvable subgroup of G; it contains some maximal torus of G. A closed subgroup of G containing some Borel subgroup is called parabolic.

The group G is called isotropic over k if some proper parabolic subgroup of G is defined over k; otherwise it is called anisotropic. If some Borel subgroup is defined over k, then G is called quasi-split; if every conjugacy class of parabolic subgroups contains a parabolic k-subgroup, then G is called split. A reductive k-group is isotropic if and only if its semisimple part (i.e. its commutator subgroup) contains a k-split torus.

Note that if k is finite or locally finite, then G is quasi-split and hence isotropic by [Bo], 16.6.

We recall some facts about the structure of parabolic k-subgroups, eg. see [BT1]. Let $P \subseteq G$ be a parabolic subgroup with unipotent radical $U = \mathcal{R}_u P$. A complement L of U in P is called a Levi subgroup of P. Levi subgroups in parabolic subgroups of reductive groups do always exist and are again reductive. Since U is normal in P, we have a semidirect product $P = L \ltimes U$. Moreover, if P is defined over k, then U is defined over k, and there exist Levi subgroups of P which are defined over k. If P is minimal among the parabolic k-subgroups, then the semisimple part L' of a Levi k-subgroup L is anisotropic; L' is also called the semisimple anisotropic kernel of G.

1.1 Roots and coroots Let $S \subseteq G$ be a k-torus. The character group $X^*(S) = \mathrm{Mor}(S, G_m)$ is free abelian and denoted additively. Dually, we consider the group $X_*(S) = \mathrm{Mor}(G_m, S)$ of all one-parameter subgroups in S. There is a pairing

$$\langle\ ,\ \rangle : X^*(S) \times X_*(S) \longrightarrow \mathbb{Z}$$

which is defined for $\alpha \in X^*(S), \beta \in X_*(S)$ by $\alpha \circ \beta(x) = x^{\langle \alpha, \beta \rangle}$ for all $x \in G_m$.

Let H be an S-invariant subgroup of G. Then S acts on the Lie algebra $\mathfrak{h}$ of H, and the weight spaces of $\mathfrak{h}$ with respect to S are $\mathfrak{h}_\alpha = \{X \in \mathfrak{h} |\ s \cdot X = \alpha(s)X \text{ for all } s \in S\}$ for $\alpha \in X^*(S)$. If $\alpha \neq 0$ and $\mathfrak{h}_\alpha \neq \{0\}$, then α is called a root; the set of all roots is denoted by $\Phi(S, H)$. There are two important root systems in G, the absolute root system $\Phi = \Phi(T, G)$, where T is a maximal k-torus, and the