MODERATE DEVIATIONS FOR FUNCTIONAL U-PROCESSES

PETER EICHELSBACHER

Abstract. The moderate deviations principle is shown for the partial sums processes built on U-empirical measures of Polish space valued random variables and on U-statistics of real valued kernel functions. It is proved that in the non-degenerate case the conditions for the time fixed principles suffice for the moderate deviations principle to carry over to the corresponding partial sums processes. Given a uniformly bounded VC subgraph class of functions, we obtain corresponding moderate deviations for time dependent U-processes. Among other techniques we use the decoupling technique for U-statistics due to de la Peña and Montgomery-Smith. We apply an improved version of a Bernstein-type inequality for degenerate U-statistics due to Arcones and Giné as well as some moment inequalities for U-statistics. Moreover we prove and use a Lévy-type maximal inequality for U-statistics.

1. Introduction and Statement of the results

For a sequence of $\mathbb{R}^d$-valued i. i. d. random variables X_i with a finite moment generating function Borovkov and Mogulskii [6] investigated the moderate deviation behaviour of the partial sums process

$$S_n(t) = \frac{1}{n} \sum_{i=1}^{[nt]} X_i, \quad t \in [0,1].$$

In an appropriate topology, Dembo and Zajic [9] proved a moderate deviation principle for the partial sums empirical process

$$L_n(t) = \frac{1}{n} \sum_{i=1}^{[nt]} \delta_{X_i}, \quad t \in [0,1].$$

Given a class of bounded functions $\mathcal{F}$, Dembo and Zajic [10] considered the moderate deviation principles for functional empirical processes

$$L_n(t,f) = \frac{1}{n} \sum_{i=1}^{[nt]} f(X_i), \quad t \in [0,1], \ f \in \mathcal{F}.$$

The aim of this paper is to extend the moderate deviation principles when passing from linear statistics to higher order statistics.

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Let us recall the definition of the large deviations principle (LDP). A sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ on a topological space $\mathcal{X}$ equipped with σ-field $\mathcal{B}$ is said to satisfy the LDP with speed $a_n \downarrow 0$ and good rate function $I(\cdot)$ if the level sets $\{x : I(x) \le \alpha\}$ are compact for all $\alpha < \infty$ and for all $\Gamma \in \mathcal{B}$ the lower bound

$$\liminf_{n\to\infty} a_n \log \mu_n(\Gamma) \ge - \inf_{x\in \mathrm{int}(\Gamma)} I(x),$$

and the upper bound

$$\limsup_{n\to\infty} a_n \log \mu_n(\Gamma) \le - \inf_{x\in \mathrm{cl}(\Gamma)} I(x)$$

hold, where $\mathrm{int}(\Gamma)$ and $\mathrm{cl}(\Gamma)$ denote the interior and closure of Γ, respectively. We say that a sequence of random variables satisfies the LDP when the sequence of measures induced by these variables satisfies the LDP. Let $\{b_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ be a sequence satisfying

$$\lim_{n\to\infty} \frac{b_n}{n} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{n}{b_n^2} = 0. \tag{1.1}$$

If $a\,x \in \mathcal{X}$ for all $a > 0$ and $x \in \mathcal{X}$ then a sequence of random variables Z_n shall satisfy the moderate deviations principle (MDP) with speed $\frac{n}{b_n^2}$ and with good rate function $I(\cdot)$, if the sequence $\frac{n}{b_n} Z_n$ satisfies the LDP in $\mathcal{X}$ with the good rate function $I(\cdot)$ and with speed $\frac{n}{b_n^2}$.

Denote by $L_\infty([0,1],\mathbb{R}^d)$ the space of (equivalence classes modulo equality a.e. of) bounded measurable functions on $[0,1]$, equipped with the uniform topology. Consider the polygonal approximation of $S_n(\cdot)$, that is

$$\widetilde{S}_n(t) := S_n(t) + \left(t - \frac{[nt]}{n}\right) X_{[nt]+1}.$$

Note that $\widetilde{S}_n(\cdot)$ is continuous and it carries the same information as $S_n(\cdot)$. The MDP for $\widetilde{S}_n(\cdot)$ was established in $L_\infty([0,1],\mathbb{R}^d)$. By $\mathcal{AC}_0([0,1],\mathbb{R}^d)$ denote the subspace of absolutely continuous functions ϕ on $[0,1]$ with $\phi(0) = 0$. Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of i. i. d. random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathbb{R}^d$, common law μ and $\mathbb{E}(X_1) = 0$. Borovkov and Mogulskii [6] considered (essentially) the MDP in the i.i.d. case under the condition that

$$\log \mathbb{E} \exp(\langle \theta, X_1 \rangle) < \infty \tag{1.2}$$

in some ball centered in the origin. $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^d$ and $\|\cdot\|$ a norm in $\mathbb{R}^d$. The sequences $\frac{n}{b_n}\widetilde{S}_n(\cdot)$ satisfies the LDP in $L_\infty([0,1],\mathbb{R}^d)$, equipped with the uniform topology, with good rate function

$$I_\infty(\phi) = \int_0^1 \Lambda^*(\dot{\phi}) dt, \tag{1.3}$$

if $\phi \in \mathcal{AC}_0([0,1],\mathbb{R}^d)$ and $I_\infty(\phi) = \infty$ otherwise. Here Λ^* denotes the convex dual of $\Lambda(\theta) = \mathbb{E}(\langle \theta, X_1 \rangle^2)/2$ (see also [21, Theorem 1] and [5, Theorem 3.1]). The MDP for $\{S_n(\cdot), n \in \mathbb{N}\}$ follows from the following