Sensitivity of the solutions of some matrix equations

Rajendra Bhatia*
Indian Statistical Institute
Delhi Center
7, S.J.S. Sansanwal Marg
New Delhi 110016
India

Ludwig Elsner
Universität Bielefeld
Fakultät für Mathematik
Postfach 10 01 31
33501 Bielefeld
Germany

November 21, 1997

Abstract

We develop a general method for obtaining bounds for the sensitivity of the solution of the stable Lyapunov equation. Our arguments not only work simultaneously for both the continuous and the discrete time Lyapunov equations, they work equally well for more general equations that have been studied. Further, we obtain bounds valid for all unitarily invariant norms.

§ 1 Introduction

In this note we consider the question of the sensitivity of the solution of some linear matrix equations. The equations we consider first are the continuous–time Lyapunov equation and the discrete–time Lyapunov equation. These are

$$A^*X + XA = W, \tag{1.1}$$

and

$$X - F^*XF = Q. \tag{1.2}$$

*This author thanks Sonderforschungsbereich 343 "Diskrete Strukturen in der Mathematik" at the University of Bielefeld for its support and hospitality during October/November 1997 when this work was done

We will assume that the matrix A is **positively stable**; i.e., all the eigenvalues of A lie in the open right half plane, and that the matrix F is **stable**; i.e., all the eigenvalues of F lie in the open unit disk. The equation (1.2) is also known as the Stein equation. Both the equations have been studied extensively because of their importance in differential equations and control theory. See, e.g. [11],[18]. More general equations have also been of interest. One that we will consider is

$$A^*X + XA + tA^{*1/2}XA^{1/2} = W, \tag{1.3}$$

where A is again positively stable, $A^{1/2}$ is its positively stable square root, and t is a real number (subject to some restrictions mentioned later). See [17] and the references therein for work on such equations.

Another equation of interest is the Sylvester equation

$$AX - XB = Y, \tag{1.4}$$

where A and B are any two matrices that have no eigenvalue in common. This equation includes the Lyapunov equation (1.1) as a special case, and has also been studied in great detail; see [6] for a recent survey.

In each of the equations (1.1)–(1.4), the matrix X is unknown and the other matrices are assumed to be given. The conditions that we have imposed ensure in each case the existence of a unique solution X. However, when the solution is computed numerically, the given matrices are subjected to perturbations. It is then of interest to obtain error bounds for the solution.

This problem too has been studied in detail. The present note supplements the existing literature, especially the work in [9], [14], [10], [8]. We make several observations that lead to a unification of some of the existing proofs, clarify some of the issues and bring in new methods that have not been employed in this context until very recently [2].

§ 2 The general framework

Consider the linear equation

$$Lx = w, \tag{2.1}$$

where x and w are vectors in a Banach space X, and L is an invertible bounded linear operator on X. Consider also the perturbed version of this equation

$$(L + \Delta L)\ (x + \Delta x) = w + \Delta w, \tag{2.2}$$

where ΔL and Δw represent small changes in L and w respectively. From these two equations we obtain

$$\Delta x = -L^{-1}\Delta L(x + \Delta x) + L^{-1}\Delta w.$$

Hence,

$$||\Delta x|| \leq ||L^{-1}||\ ||\Delta L||\ ||x + \Delta x|| \ + ||L^{-1}||\ ||\Delta w||,$$