PARTIAL NORMS AND THE CONVERGENCE OF GENERAL PRODUCTS OF MATRICES

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Abstract

Motivated by the theory of inhomogeneous Markov chains, we determine a sufficient condition for the convergence to 0 of a general product formed from a sequence of real or complex matrices. When the matrices have a common invariant subspace H, we give a sufficient condition for the convergence to 0 on H of a general product. Our result is applied to obtain a condition for the weak ergodicity of an inhomogeneous Markov chain. We compare various types of contractions which may be defined for a single matrix, such as paracontraction, l–contraction, and H–contraction, where H is an invariant subspace of the matrix.

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1 Introduction

Recently there has been much interest in conditions for the convergence of infinite products of real or complex matrices. Several investigations have concentrated on products taken in one direction – left or right, see for example the recent papers by Beyn and Elsner [2] and Hartfiel and Rothblum [4]. However, in this paper, we are concerned with *general products* formed from a given infinite sequence of matrices. These are defined further on in the paper and they have previously been considered for nonnegative and for stochastic matrices by Seneta in [16, Chapters 3.1 and 4.6].

Our principal result is a sufficient condition for the convergence to 0 of infinite general products of matrices. We pay particular attention to the case where there is a common invariant subspace for all the matrices in the product. As a special case, we obtain a result on the weak ergodicity of an inhomogeneous Markov chain.

The investigations described above are preceded by a study of the interrelations of several types of contractions which may be defined for a single matrix.

The motivation for our study comes from the theory of inhomogeneous Markov chains, see [15] and [16, Chapter 4] for much further background material. This theory naturally leads to the study of the convergence, ergodicity and weak ergodicity of products of stochastic matrices.

We now describe our paper in more detail. First, in Section 2, we sharpen several observations concerning paracontraction. This we shall achieve by introducing a partial norm on the matrix restricted to an invariant subspace H and the consequent notion of *H–contraction.* We relate this concept to the notion of paracontraction as introduced by Nelson and Neumann in [13] and to the notion of l–contraction recently introduced by Beyn and Elsner in [2].

Second, in Section 3, which is essentially independent of Section 2, we turn to our main results. We formulate our sufficient conditions depending on norms for the convergence to 0 of infinite general products of matrices which are formed from a given infinite sequence of matrices. We then apply