ENUMERATING REPRESENTATIONS IN FINITE WREATH PRODUCTS

THOMAS MÜLLER[†]

ABSTRACT. Let Γ be a group (finite or infinite), H a finite group, and let R_n denote the sequence $H \wr S_n$ of symmetric wreath products as well as certain variants of it (including in particular $H \wr A_n$ and W_n, the Weyl group of type D_n) . We compute the exponential generating function for the number $|\mathrm{Hom}(\Gamma, R_n)|$ of Γ-representations in R_n and for some refinements of this sequence under very mild finiteness assumptions on Γ (always met for instance if Γ is finitely generated). This generalizes in a uniform way the connection between the problem of counting finite index subgroups in a group Γ and the enumeration of Γ-actions on finite sets on the one hand, and the recent results of Chigira concerning solutions of the equation $x^m = 1$ in the groups $H \wr S_n$, $H \wr A_n$, and W_n on the other. We also study the asymptotics of the function $|\mathrm{Hom}(G, H \wr S_n)|$ for arbitrary finite groups G and H.

INTRODUCTION

For a group Γ denote by $s_\Gamma(n)$ the number of subgroups of index n in Γ. If Γ is finitely generated or of finite (subgroup) rank then $s_\Gamma(n)$ is finite for all n. There is a well-known connection between the problem of counting finite index subgroups in a group Γ and the enumeration of Γ-actions on finite sets (i.e. permutation representions of Γ), manifesting itself in a variety of results scattered throughout the literature. One of the earliest results in this direction is M. Hall's recursion formula

$$s_{F_r}(n) = n(n!)^{r-1} - \sum_{k=1}^{n-1} \big((n-k)!\big)^{r-1} s_{F_r}(k)$$

for the number of index n subgroups in the free group of (finite) rank r; cf. [8, Theorem 5.2]. Dey's corresponding formula [4, Theorem 6.10] for (non-trivial) free products is another relevant example. Denote by $\beta_m(B)$ the number of solutions of the equation $x^m = 1$ in a finite group B. Chowla, Herstein and Scott [3] obtain the exponential generating function of the sequence $\{\beta_m(S_n)\}_{n=0}^\infty$ as

$$\sum_{n=0}^{\infty} \frac{\beta_m(S_n)}{n!} z^n = \exp\left(\sum_{d|m} \frac{z^d}{d} \right).$$

The more general relation

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(G, S_n)|}{n!} z^n = \exp\left(\sum_{d|m} \frac{s_G(d)}{d} z^d \right), \quad m = |G|$$

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for an arbitrary finite group G, which exhibits the exponential generating function of the sequence $\{|\mathrm{Hom}(G, S_n)|\}_0^\infty$ as a particular type of entire function, was a starting point in [14] for the asymptotic enumeration of finite group actions; cf. [14, Theorem 5]. The most general expression of this correspondence to be found in the literature appears to be [7, Prop. 1], which apart from incorporating the results mentioned, can also be used for example to count torsion-free subgroups of finite index. This latter aspect has served as a starting point for a detailed analysis of the growth behaviour and the asymptotics of the number of free subgroups of finite index in a finitely generated virtually free group; cf. [12] and [13].

The subject gained an interesting new perspective with the appearance of Chigira's paper [1]. In this paper Chigira studies the equation $x^m = 1$ in finite wreath products of the form $H \wr S_n$ and $H \wr A_n$, and in the Weyl groups W_n of type D_n, obtaining generating functions for the sequences $\{\beta_m(H \wr S_n)\}_{n=0}^\infty$, $\{\beta_m(H \wr A_n)\}_{n=0}^\infty$, and $\{\beta_m(W_n)\}_{n=0}^\infty$; cf. [1, Theorems 2, 4, and 5]. His result for the representation sequence $\{H \wr S_n\}$ for instance is that

$$\sum_{n=0}^\infty \frac{\beta_m(H \wr S_n)}{n!} z^n = \exp\left(\sum_{d|m} \frac{|H|^{d-1}\beta_{m/d}(H)}{d} z^d\right).$$

Furthermore, Chigira indicates how to modify Wilf's argument in [20] leading to an asymptotic formula for $\beta_m(S_n)$ with m fixed and $n \to \infty$ to obtain a similar asymptotic formula for $\beta_m(H \wr S_n)$; cf. [1, Theorem 3].
The purpose of the present paper is threefold: (i) to generalize Chigira's results [1, Theorems 2, 4, 5] from the consideration of cyclic groups to the enumeration of wreath product representations of (almost) arbitrary groups Γ, (ii) to replace Chigira's case-by-case study of the sequences $\{H \wr S_n\}$, $\{H \wr A_n\}$, and $\{W_n\}$ by an approach working uniformly for these and certain other representation sequences, and (iii) to study in detail the asymptotics of the function $|\mathrm{Hom}(G, H \wr S_n)|$ for arbitrary finite groups G and H.
Apart from their intrinsic interest, our results also provide a key ingredient for a substantial improvement over the results obtained in [15] concerning the subgroup growth of virtually free groups. This development ultimately leads to a solution of Poincaré's problem concerning the asymptotic distribution of the isomorphism types of finite index subgroups in the modular group (and other virtually free groups). These aspects will be explained in forthcoming work; cf. [16] and [17].

Let $\{\Pi_n\}$ denote either the sequence $\{A_n\}$ of alternating groups or the sequence $\{S_n\}$ of symmetric groups. Moreover, given $\{\Pi_n\}$, a finite group H, and a normal subgroup $N \trianglelefteq H$ with H/N abelian put

$$(H \wr \Pi_n)_N := \Big\{(f, \pi) \in H \wr \Pi_n : \prod_i f(i) \in N\Big\}, \quad n \geq 0.$$

Our main counting result (Theorem 1) computes the exponential generating function of the sequence $\{|\mathrm{Hom}(\Gamma, (H \wr \Pi_n)_N)|\}_{n=0}^\infty$ and of certain refinements of it whenever $(H : N) \leq 2$ and Γ satisfies some very mild finiteness assumptions (always met for instance if Γ is finitely generated).