Metrics with finite sets of primitive extensions

Alexander V. KARZANOV ♮

Abstract. This paper gives a complete characterization of the class of rational-valued finite metrics μ such that the set $\Pi(\mu)$ of primitive extensions of μ is finite. More precisely, for a metric μ on a set T of points, an *extension* of μ is a (semi)metric m on a set $V \supseteq T$ which coincides with μ within T, and m is said to be *primitive* if $m(xy) \neq 0$ for all distinct $x, y \in V$ and m is a vertex of the dominant of the polyhedron formed by all extensions of μ to V.

As the main result, we show that $\Pi(\mu)$ is finite if and only if, for some integer $\lambda > 0$, $\lambda\mu$ is a submetric of the path metric d^H of a so-called *minimizable* graph H (such graphs come up in connection with a generalization of the minimum multi-terminal cut problem and are exactly those graphs H whose metric d^H has no primitive extensions except itself). Moreover, we explicitly construct such an H, explain that the finiteness of $\Pi(\mu)$ can be recognized efficiently, show that $\Pi(\mu)$ is finite if and only if the tight span of μ is 2-dimensional, and give other results. Our results essentially rely on properties of minimizable graphs described in [Ka97].

Key words: Finite metric, Metric extension, Isometric embedding, Multi-terminal cut, Tight span.

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1. Introduction

This paper continues our study of finite metrics from the viewpoint of their primitive extensions begun in [9] where necessary and sufficient conditions for a graph metric that admits a unique primitive extension are described. Here we give a complete characterization of the set of finite metrics that may have more than one but a finite number of primitive extensions, thus answering the question raised in [8].

Throughout by a *metric* on a set V' we mean a nonnegative real-valued function m' that establishes *distances* of the pairs of elements of V' satisfying (i) $m'(x,x)=0$, (ii) $m'(x,y)=m'(y,x)$, and (iii) $m'(x,y)+m'(y,z)\geq m'(x,z)$, for all $x,y,z\in V'$. Unless otherwise is said, we assume that V' is finite, allow zero distances between different elements (i.e., m' is, in fact, a semimetric), and call a metric m' *positive* if $m'(x,y)>0$ for all distinct $x,y\in V'$. We do not distinguish between the metric m' and metric space (V',m'); elements of V' are called *points* of this space. Because of (i) and (ii), it suffices to define m' on the set $E_{V'}$ of unordered pairs of distinct elements of V', or, equivalently, on the edge set of the complete (undirected) graph $K_{V'}=(V',E_{V'})$. We write xy and $m'(xy)$ in place of $\{x,y\}$ and $m'(x,y)$, respectively.

We deal with a positive rational-valued metric μ on a set T. A metric m on $V\supseteq T$ satisfying $m(st)=\mu(st)$ for all $s,t\in T$ is called an *extension* of μ to V. In other words, m is an extension of μ if μ is a submetric of m, denoted as $\mu=m_{|T}$. If m admits no other extensions m',m'' such that $m\geq\lambda m'+(1-\lambda)m''$ for some $0\leq\lambda\leq 1$, then m is called *extreme*. In other words, the extreme extensions of μ to V are exactly the vertices of the dominant $\mathcal{D}(\mu,V)$ of the polyhedron $\mathcal{P}(\mu,V)$ formed by the extensions of μ to V. (The *dominant* of a set $X\subseteq\mathbb{R}^n$ is $\{z\in\mathbb{R}^n : z\geq z' \text{ for some } z'\in X\}$.) A positive extreme extension is called *primitive*.

Clearly if $m(xy)=0$ for some $x,y\in V$, then $m(xz)=m(yz)$ for all $z\in V$. Therefore, shrinking each maximal subset of points with zero distances between them into a single point makes a positive metric m' on the factor set; moreover, it is easy to see that m is an extreme extension of μ if and only if m' is a primitive extension of μ. We call an extension m of μ a *0-extension* if each point of V is at zero distance from some point of T; in other words, the above shrinking for m produces μ. So every 0-extension is extreme.

Let $\Pi(\mu)$ denote the set of primitive extensions of μ (regarding all finite sets $V\supseteq T$). Note that some metrics μ have infinitely many primitive extensions. We are interested in the case when $\Pi(\mu)$ is finite. Our description of such metrics involves so-called minimizable graphs.

Originally the concept of minimizability came up in connection with some generalization of the multi-terminal cut problem (a special case of the latter problem is the

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Abstract. This paper gives a complete characterization of the class of rational-valued finite metrics μ such that the set $\Pi(\mu)$ of primitive extensions of μ is finite. More precisely, for a metric μ on a set T of points, an *extension* of μ is a (semi)metric m on a set $V \supseteq T$ which coincides with μ within T, and m is said to be *primitive* if $m(xy) \neq 0$ for all distinct $x, y \in V$ and m is a vertex of the dominant of the polyhedron formed by all extensions of μ to V.

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Let $\Pi(\mu)$ denote the set of primitive extensions of μ (regarding all finite sets $V\supseteq T$). Note that some metrics μ have infinitely many primitive extensions. We are interested in the case when $\Pi(\mu)$ is finite. Our description of such metrics involves so-called minimizable graphs.

Originally the concept of minimizability came up in connection with some generalization of the multi-terminal cut problem (a special case of the latter problem is the classical minimum cut problem in network theory). For a connected graph $H=(W,U)$, let d^H denote its *distance function*, or *path metric* (regarding the all-unit lengths of edges), i.e., $\mathrm{d}^H(xy)$ is the minimum number of edges of a path in G connecting nodes x and y. Following [10], H is called *minimizable* if for any set $V \supseteq W$ and function $c : E_V \to \mathbb{Z}_+$, the minimum objective value in the problem $\min\{cm : m$ is a 0-extension of d^H to $V\}$ is equal to that in its relaxation $\min\{cm : m$ is an extension of dH to $V\}$. In our terms, this is equivalent to saying that H is minimizable if d^H has no primitive extensions except d^H itself.

Our main result in this paper is the following.

Theorem 1.1. *Let μ be a rational positive metric on a finite set T. Then $\Pi(\mu)$ is finite if and only if there exist a minimizable graph H and a positive integer λ such that $\lambda\mu$ is a submetric of* d^H.

The class of metrics μ with $\Pi(\mu)$ finite can be characterized more explicitly by use of a construction that we now describe. For a metric m on V, a point $v \in V$ is called a *median* of a triple $\{s_0, s_1, s_2\}$ in V if

$$m(s_i v) + m(v s_j) = m(s_i s_j) \qquad \text{for all } 0 \le i < j \le 2. \tag{1.1}$$

We construct a certain extension m of μ by the following process. Initially set $V := T$ and $m := \mu$. Choose in V a triple $\{s_0, s_1, s_2\}$ without a median, add a new point v to V and define the distances from v to the s_i's so as to satisfy (1.1) (such distances exist and are unique). Then we define distances from v to the other points in V as follows. Let $V' \subset V$ be the set of points of which distances from v have already been defined; initially $V' = \{s_0, s_1, s_2, v\}$. Choose an arbitrary $u \in V - V'$ and put

$$m(uv) := \max\{m(ux) - m(xv) \; : \; x \in V' - \{v\}\}. \tag{1.2}$$

Set $V' := V' \cup \{u\}$ and iterate until $V' = V$. One can check that m remains a metric and an extension of μ. Repeat the procedure for a next medianless triple $\{s_0, s_1, s_2\}$ for the current V and m, and so on. When the process terminates, the resulting (V, m) has a median for each triple and is the desired extension of μ. We call m (or (V, m)) obtained this way a *median closure* of μ. Note that m depends on the order in which triples in T are treated, so median closures corresponding to different orders may a priori differ.

One shows that for a rational metric μ the above process does terminate in a finite number of steps (Statement 2.1). Let $G = (V, E)$ be the graph obtained from K_V by deleting all edges xy such that there is a node $z \in V - \{x, y\}$ *between* x and y, i.e., satisfying $m(xz) + m(zy) = m(xy)$. Then m coincides with the path metric in $(G, m_{|E})$, i.e., for any $x, y \in V$, $m(xy)$ is equal to the minimum length of a path connecting x

and y in G, letting the length of an edge $e \in E$ be $m(e)$. Such a G is called the *least graph generating* m. We prove the following.

Theorem 1.2. *Let μ be a rational positive metric on a finite set T. Let (V, m) be a median closure of μ, and let $G = (V, E)$ be the least graph generating m. Then the following are equivalent:*

(i) $\Pi(\mu)$ is finite;

(ii) G is a hereditary modular graph without induced subgraphs $K^-_{3,3}$.

Hereinafter we use the following terminology and notations. A subgraph $G' = (V', E')$ of G is *induced* if any two nodes $x, y \in V'$ are adjacent (i.e., connected by an edge) in G' when they are so in G, and *isometric* if $\mathrm{d}^{G'}(xy) = \mathrm{d}^G(xy)$ for all $x, y \in V'$. A graph G is called *modular* if every triple of nodes of G has a median (w.r.t. the metric d^G), and *hereditary modular* if every isometric subgraph of G is modular, cf. [3]. It is easy to see that any modular graph is bipartite. $K^-_{3,3}$ is the graph obtained by deleting one edge from $K_{3,3}$ (where $K_{p,q}$ is the complete bipartite graph whose parts consist of p and q nodes); see Fig. 1. We call a graph G as in (ii) of Theorem 1.2 a *semiframe* (the meaning of this term will be clearer later).

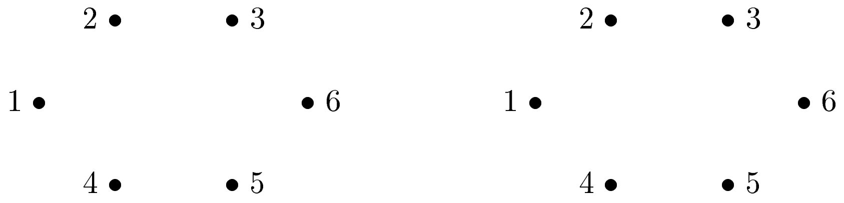

Fig. 1 (a) $K_{3,3}$ (b) $K^-_{3,3}$

The next result concerns so-called tight spans of metrics (also known in literature as injective envelopes, T_X-spaces, universal tight extensions). An extension m' of μ to a set V' is called *tight* if there is no $m'' \in \mathcal{P}(\mu, V') - \{m'\}$ such that $m'' \leq m'$. Isbell [7] and Dress [6] showed that any metric space (X, d) can be uniquely extended to a minimal metric space $(\mathcal{X}, \delta)$ such that any tight extension (X', d') of (X, d) can be isometrically embedded in $(\mathcal{X}, \delta)$, i.e., there exists a mapping $\gamma : X' \to \mathcal{X}$ satisfying $\gamma(x) = x$ for all $x \in X$ and $\delta(\gamma(x)\gamma(y)) = d'(xy)$ for all $x, y \in X'$. Such an $(\mathcal{X}, \delta)$ is just the *tight span* of (X, d). When X is finite, $\mathcal{X}$ can be represented as a simplicial complex of dimension $\leq |X|/2$. We prove the following.

Theorem 1.3. $\Pi(\mu)$ *is finite if and only if the dimension of the tight span $\mathcal{T}(\mu)$ of μ is at most two.*

The method of proof of the above theorems essentially relies on several results on minimizable graphs established in [9]. Two of them, exhibited in the next theorem, are most important for us.

Theorem 1.4 [9]. *Let H be a connected graph.*

(i) H is minimizable if and only if H is bipartite, orientable and contains no isometric k-cycle with $k \geq 6$.

(ii) If H contains an isometric 6-cycle or an induced $K^-_{3,3}$, then there exists a non-integral primitive extension m' of d^H; moreover, m' has a submetric m'' on six points such that $m'' = \frac{1}{2}\mathrm{d}^{K^-_{3,3}}$.

Here a *k-cycle* is a (simple) circuit C_k on k nodes (considered as a closed path or as a graph depending on the context). A graph H is *orientable* if the edges of H can be oriented so that for any 4-cycle $C = (v_0, e_1, v_1, ..., e_4, v_4 = v_0)$, the orientation of the opposite edges e_1 and e_3 are opposite along the cycle, and similarly for e_2 and e_4 (a feasible orientation is depicted in Fig. 2).

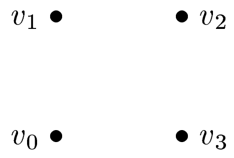

Fig. 2

Bandelt [3] showed that the bipartite graphs without isometric k-cycles with $k \geq 6$ are exactly the hereditary modular graphs (see Theorem 1.9 below). Thus, H is minimizable if and only if it is hereditary modular and orientable. [9] refers to such an H as a *frame*. Note that the graph $K^-_{3,3}$ is non-orientable. Hence, every frame is a semiframe (defined above); the converse is not, in general, true.

Statement (ii) in Theorem 1.4 will enable us to recursively construct an unbounded sequence of primitive extensions of μ in the case when G as above is not a semiframe, thus proving part (i)→(ii) in Theorem 1.2. The reverse part (ii)→(i) is more involved and it is based on statement (i) in Theorem 1.4 and a so-called "orbit splitting method" elaborated in Section 4. The idea is roughly as follows.

First of all note that if μ' is a metric on T such that $\mu' = \lambda\mu$ for some $\lambda > 0$, then m is a primitive extension of μ if and only if λm is a primitive extension of μ'. Thus, we can consider μ up to proportionality (with a positive factor) without affecting the problem in question. We call μ *cyclically even* if the μ-length of any cycle on T is an even integer (or, equivalently, $\mu(xy) + \mu(yz) + \mu(zx)$ is even for all $x, y, z \in T$); in particular, $\mu(xy)$ is an integer for all $x, y \in T$ (since $\mu(xy) + \mu(yy) + \mu(yx)$ is even). In what follows we assume w.l.o.g. that μ is cyclically even.

If G is already a frame and $m(e) = 1$ for all $e \in E$, then m is just d^G, and the fact that d^G has a unique primitive extension (namely, d^G itself) provides that every primitive extension of μ corresponds to a submetric of d^H, implying the finiteness of

$\Pi(\mu)$. In a general case, we consecutively split orbits of G (where an *orbit* is meant to be a component of the graph whose nodes correspond to the edges of G and the edges correspond to the pairs of edges of G which are opposite in 4-cycles of G). A finite number of orbit splittings transforms G into a larger graph G' such that G' is a frame and $\frac{1}{2}\mathrm{d}^{G'}$ is a tight extension of μ. This provides the finiteness of $\Pi(\mu)$, yielding (ii)→(i) in Theorem 1.2. Also this construction together with Theorem 1.2 will imply Theorem 1.1. Furthermore, as a by-product of our approach we obtain the following.

Corollary 1.5. *If μ is cyclically even and $\Pi(\mu)$ is finite, then every primitive extension of μ is half-integer.*

Next, [9] gives an explicit combinatorial construction of the tight span $\mathcal{T}(\mathrm{d}^H)$ when H is a frame, showing that $\mathcal{T}(\mathrm{d}^H)$ is 2-dimensional (unless H is a tree). This fact is used as the main ingredient in the proof of Theorem 1.3. Moreover, we obtain the following result, which seems to be interesting in its own right.

Corollary 1.6. *Let μ be a rational finite metric whose tight span has the dimension at most two. Then there are a frame H and an integer $\lambda > 0$ such that $\mathcal{T}(\lambda\mu)$ is isomorphic to $\mathcal{T}(\mathrm{d}^H)$. In other words, up to proportionality, the set of 2-dimensional tight spans of rational metrics is exactly the set of tight spans of path metrics of frames (different from trees).*

It turns out that the metrics μ with $|\Pi(\mu)| < \infty$ can also be characterized in local terms. More precisely, Dress found two interesting local properties of metrics.

Theorem 1.7 [6]. *For a metric space (X, d),*

(i) if $\mathcal{T}(d)$ is k-dimensional ($k < \infty$), then there is a submetric d' of d on $2k$-points such that $\mathcal{T}(d')$ is k-dimensional;

(ii) if $|X| = 2k$, then $\mathcal{T}(d)$ is k-dimensional if and only if there is a perfect matching M in K_X such that $\sum(d(e)\,:\, e \in M) > \sum(d(e)\,:\, e \in M')$ holds for all other perfect matchings M' in K_X.

As an immediate consequence of (i) and Theorem 1.3, we have the following local criterion.

Corollary 1.8. *$\Pi(\mu)$ is finite if and only if $\Pi(\mu')$ is finite for every submetric μ' of μ on six points.*

In its turn, (ii) in Theorem 1.7 shows that, given a μ, the problem of deciding whether $\Pi(\mu)$ is finite or not is solvable in strongly polynomial time. Indeed, we can simply enumerate all six-element subsets T' of T, and for each such T', enumerate all three-edge matchings in $K_{T'}$ and check whether the μ-length of one of these is strictly greater than the μ-length of each of the others.

Among other tools in our proofs we use results of Bandelt on hereditary modular graphs. In particular, he proved the following facts important to us.

Theorem 1.9. [3]. *Let* $H = (T, U)$ *be a graph.*

(i) H *is hereditary modular if and only if* H *is bipartite and contains no isometric* k*-circuit with* $k \geq 6$.

(ii) If H *is modular but not hereditary modular, then* H *contains an isometric 6-circuit, which, in its turn, is contained in a (not necessarily induced) cube in* H *(see Fig. 3).*

(iii) If H *is bipartite but not modular, then* H *contains a medianless triple* $\{s_0, s_1, s_2\}$ *with* $\mathrm{d}^H(s_0 s_1) = \mathrm{d}^H(s_0 s_2) \geq 2$ *and* $\mathrm{d}^H(s_1 s_2) = 2$.

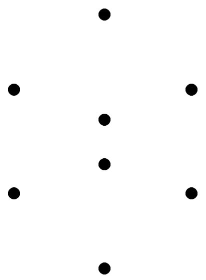

Fig. 3

This paper is organized as follows. Section 2 justifies the process of constructing a median closure m of μ and demonstrate a relationship between shortest paths for m and for G. The main goal of Section 3 is to prove part (i)→(ii) of Theorem 1.2. Section 4 describes the orbit splitting method, shows how to prove, with its help, the reverse part (ii)→(i) of Theorem 1.2, and explains how to obtain Theorems 1.1 and 1.3 and Corollaries 1.5 and 1.6.

2. Backgrounds

As mentioned in the Introduction, we may assume that μ be a positive cyclically even metric on T. For an extension m of μ to V, a sequence $P = (x_0, x_1, \ldots, x_k)$ of points of V is called a *path on* V, and a T*-path* if $x_0, x_k \in T$. For brevity we write $P = x_0 x_1 \ldots x_k$. A closed path (i.e., with $x_0 = x_k$) is a *cycle*. The *length* of P with respect to m, or the m*-length*, is $m(P) = m(x_0 x_1) + \ldots + m(x_{k-1} x_k)$, and P is called *shortest* w.r.t. m, or m*-shortest*, if $m(P) = m(x_0 x_k)$. The set of shortest T-paths is denoted by $\mathcal{G}(m)$.

For $m, m' \in \mathcal{P}(\mu, V)$, we say that m' *decomposes* m if $m \geq \lambda m' + (1 - \lambda) m''$ for some $m'' \in \mathcal{P}(\mu, V)$ and $0 < \lambda \leq 1$; so m is extreme if and only if no $m' \neq m$ decomposes m. It is easy to see that m' decomposes m if and only if $\mathcal{G}(m) \subseteq \mathcal{G}(m')$.

We will also use a simple characterization of tight extensions in terms of shortest paths (see, e.g., [7]): an extension m of μ to V is tight if and only if

(2.1) for any $x, y \in V$, there are $s, t \in T$ such that $m(sx) + m(xy) + m(yt) = m(st)$ $(= \mu(st))$, i.e., x, y are contained in an m-shortest T-path.

First of all we have to show that the construction of a median closure (V, m) for μ described in the Introduction is correct.

Statement 2.1. *The process of constructing a median closure (V, m) terminates in a finite number of iterations. Moreover, m is cyclically even and primitive.*

Proof. Suppose that after a number of iterations we have obtained a cyclically even primitive extension m on a current set V and that the next iteration chooses a medianless triple $\{s_0, s_1, s_2\}$ and add a median v for it. We observe from (1.1) that $m(s_0v)$ is uniquely determined to be $\frac{1}{2}(m(s_0s_1) + m(s_0s_2) - m(s_1s_2))$, and similarly for $m(s_1v)$ and $m(s_2v)$; so the numbers $m(s_iv)$ are positive integers. Moreover, the submetric of m on $V^0 = \{s_0, s_1, s_2, v\}$ is, obviously, cyclically even.

Let $V - V^0$ consist of the points $s_3, \ldots, s_n$ which are chosen in this order when the distances from v to these points are determined. By rule (1.2), for each $i = 3, \ldots, n$, there is $j < i$ such that $m(s_iv) + m(vs_j) = m(s_is_j)$, i.e., the path $P_i = s_ivs_j$ is m-shortest. Then (by induction on i) $m(s_iv)$ is an integer and the m-length of the cycle $s_ivs_js_i$ is even. This easily implies that the m-length of any cycle of the form $s_pvs_qs_p$, $p, q = 1, \ldots, n$ is even, and now the fact that the metric on $\{s_0, \ldots, s_n, v\}$ is cyclically even follows from a similar property for its submetric $\widetilde{m}$ on $\widetilde{V} = \{s_0, \ldots, s_n\}$. Note also that the new m is positive (otherwise $m(vs_i) = 0$ for some i, whence s_i is a median for $\{s_0, s_1, s_2\}$).

To see that the new m is primitive, consider the paths P_i, $i = 3, \ldots, n$ as above and the paths $P_0 = s_0vs_2$, $P_1 = s_1vs_0$ and $P_2 = s_2vs_1$. Since the previous metric $\widetilde{m}$ is primitive (and therefore, tight), the ends s_i and s_j of each P_i are contained in an $\widetilde{m}$-shortest T-path ss_is_jt (cf. (2.1)), whence $P'_i = ss_ivs_jt$ is also an m-shortest T-path. Let an extension m' of μ to $\{s_0, \ldots, s_n, v\}$ decompose m. Since $\widetilde{m}$ is primitive, m and m' coincide within $\widetilde{V}$. Also each P'_i is m'-shortest (since $\mathcal{G}(m) \subseteq \mathcal{G}(m')$), whence each P_i is m'-shortest. But the system $\{P_0, \ldots, P_n\}$ of shortest paths determine the distances on $vs_0, \ldots, vs_n$ uniquely; so m and m' coincide on these pairs. Therefore, $m' = m$, yielding the primitivity of m.

Finally, since each iteration results in a tight metric m on a current V, any $s \in T$ and $x \in V - T$ belong to an m-shortest T-path sxt. This implies $m(sx) \le \max\{\mu(s't') : s', t' \in T\} =: a$. Also m is integer-valued and any two different $x, y \in V - T$ belong to an m-shortest T-path $sxyt$, whence $m(sx) \ne m(sy)$. Thus, the number of elements of

$V - T$ (i.e., the number of iterations) cannot exceed $a^{|T|}$. •

Next we demonstrate some properties of the least generating graph $G = (V, E)$ for a median closure (V, m) of μ ((2.2) and Statement 2.2 below). Such properties have been known for modular spaces [2] (see also [4] for a more general case); however, to make our paper more self-contained, we give their direct proofs. A path $P = x_0x_1 \dots x_k$ on V is a *path in* G if $x_{i-1}x_i \in E$ for $i = 1, \dots, k$; P is a *cycle in* G if $x_0 = x_k$ and all edges $x_{i-1}x_i$ are different. The number k of edges of P is denoted by $|P|$ and called the *G-length* of P. A shortest path in G is called *G-shortest.* An *s–t path* is a path with ends s, t. First of all we observe that

(2.2) every simple path P with $|P| \le 2$ in G is simultaneously m-shortest and G-shortest.

Indeed, this is obvious if $|P| = 1$. Let $|P| = 2$ and $P = xyz$. Take a median v (w.r.t. m) for $\{x, y, z\}$. If $v = y$ then P is m-shortest, whence P is G-shortest too (since y is between x and y for m). And if $v \ne y$ then, letting for definiteness that $v \ne x$, the equality $m(xv) + m(vy) = m(xy)$ shows that the edge xy is redundant in G; a contradiction.

Statement 2.2. *A path in G is m-shortest if and only if it is G-shortest.*

Proof. Consider two paths $P = sx_1 \dots x_kt$ and $Q = sy_1 \dots y_qt$ in G with the same ends such that P is G-shortest and Q is m-shortest. It suffices to show that $m(P) = m(Q)$ and $|P| = |Q|$. We use induction on $|P|$. Case $|P| = 1$ is obvious.

(i) Let $|P| = 2$, i.e., $k = 1$. By (2.2), $m(P) = m(Q)$. Suppose that $|Q| \ge 3$. Since no edge in G is redundant, each of $y_1, \dots, y_q$ is different from $x = x_1$. Take a median v for $\{s, x, y_q\}$. Then $v = s$. Indeed, if $v \ne s, x$ then the edge sx is redundant, while if $v = x$ then the edge xt is redundant (taking into account that the path svy_qt is, obviously, m-shortest). Now $v = s$ implies that the path $xsy_1 \dots y_q$ is m-shortest, whence $m(xy_q) > m(xy_1)$ (as $y_q \ne y_1$ and m is positive). Arguing similarly for the triple $\{t, x, y_1\}$, we obtain that the path $xty_q \dots y_1$ is m-shortest, whence $m(xy_1) > m(xy_q)$; a contradiction. Thus, $|Q| = 2$.

(ii) Let $|P| \ge 3$ (and therefore, $|Q| \ge 3$). If some x_i belongs to Q, the result immediately follows by induction. So assume that P and Q have no common intermediate nodes. Take a median v for $\{s, x_k, y_q\}$. Then v belongs to some m-shortest s–x_k path L' and some m-shortest s–y_q path L'' in G. One may assume that the part L from s to v is the same in these paths; let $L = sz_1 \dots z_pv$. By induction $|P'| = |L'|$ and $m(P') = m(L')$, where P' is the part of P from s to x_k. Next, let D be the m-shortest path in $L' \cup L''$ that connects x_k, y_q and passes v. Since the path x_kty_q is G-shortest (by (2.2)), we have $|D| = 2$, by (i).

Observe that v cannot coincide with either of x_k, y_q. For if $v = x_k$, consider the concatenation R of L'' and $y_q t$. Since L'' and Q are m-shortest, R is m-shortest too. But R passes x_k; hence, $x_k t$ is redundant. And if $v = y_q$ then replace in L' the part from y_q to x_k by $y_q t x_k$, forming the s–x_k path $\widetilde{L}$ which is again m-shortest. By induction $|\widetilde{L}| = |P'|$, i,e., $\widetilde{L}$ is G-shortest, whence $\mathrm{d}^G(sx_k) > \mathrm{d}^G(st)$. This is impossible because s, x_k, t follow in this order in the G-shortest path P.

Thus, $L' = sz_1 \ldots z_p v x_k$ and $L'' = sz_1 \ldots z_p v y_q$. This implies $|P'| = |L'| = |L''|$, whence the s–t path $R = sz_1 \ldots z_p v y_q t$ satisfies $|R| = |P|$, i.e., R is G-shortest. Therefore, L'' is also G-shortest. Applying induction to L'' and the part Q' of Q from s to y_q, we have $|L''| = |Q'|$, whence $|Q| = |P|$. Finally, to see that P is m-shortest, notice that the paths $vx_k t$ and $vy_q t$ are m-shortest. Then

$$\begin{aligned} m(P) = m(P') + m(x_k t) &= m(L) + m(vx_k) + m(x_k t) = m(L) + m(vy_q) + m(y_q t) \\ &= m(L'') + m(y_q t) = m(Q') + m(y_q t) = m(Q), \end{aligned}$$

as required. •

This statement provides that G is modular. Indeed, for any $s_0, s_1, s_2 \in V$, there are s_0–s_1, s_1–s_2 and s_2–s_0 paths in G which are m-shortest and share a common node v. Then these paths are G-shortest, therefore, v is a median for $\{s_0, s_1, s_2\}$ w.r.t. d^G. Another corollary from Statement 2.2 is that every isometric subgraph (or cycle) G' in G is *m-isometric*, i.e., any two nodes in G' are connected by an m-shortest path which is entirely contained in G'. Moreover, the tightness of m enables us to sharpen this property as follows:

(2.3) if $G' = (V', E')$ is an isometric subgraph (or cycle) in G, then for any $x, y \in V'$ there is an m-shortest T-path which passes x and y and whose part between these nodes is contained in G'.

3. Infinite sets of primitive extensions

Suppose that the graph $G = (V, E)$ as above is not a semiframe, i.e., it is not hereditary modular or contains an induced subgraph $K^-_{3,3}$. We show that μ has infinitely many primitive extensions, utilizing some results from [9]. One of them demonstrates a situation when extreme (or primitive) extensions can be constructed recursively.

Statement 3.1 [9]. *Let m' be an extreme extension of a metric μ to a set V'. Let μ' be a submetric of m' on some $T' \subseteq V'$. Let m'' be an extreme extension of μ' to a set V'' such that $V' \cap V'' = T'$. Then there exists an extreme extension $\widetilde{m}$ of μ to $W = V' \cup V''$ such that $\widetilde{m}$ coincides with m' on V' and with m'' on V''.*

As mentioned in the previous section, the graph G is modular; in particular, G is bipartite and, therefore, any 4-cycle in it is isometric. Consider an isometric cycle $C = v_0v_1 \dots v_{2k-1}v_0$ in G. By the argument in the previous section, C is m-isometric. Moreover, by (2.3) (for $G' = C$), for each $i = 0, \dots, k-1$, there exists an m-shortest T-path which passes opposite nodes v_i and v_{i+k}, and its subpath P between these nodes is a G-shortest path in C, i.e., P is one of the two halves of C separated by v_i and v_{i+k}. Then

$$(3.1)\quad \sum(m(v_{i+j}v_{i+j+1})\ :\ j = 0, \dots, k-1) = \sum(m(v_{i+j}v_{i+j+1})\ :\ j = k, \dots, 2k-1),$$

taking indices modulo $2k$. Putting together such equalities for $i = 0, \dots, k-1$, one can deduce that the distances of any two opposite edges in C are the same, i.e.,

$$(3.2)\qquad m(v_iv_{i+1}) = m(v_{i+k}v_{i+k+1}) \quad \text{for all } i = 0, \dots, k-1.$$

It should be noted that relations as in (3.1) and (3.2) occurred in [1,11]) and were used to prove the primitivity of certain metrics of graphs. We apply a similar approach to construct needed primitive extensions. Following [11], edges e, e' in G are called *dependent* if there is a sequence $e = e_0, e_1, \dots, e_p = e'$ of which each two consecutive edges e_j, e_{j+1} are opposite in some (even) isometric cycle in G. Then the distances of such e, e' are the same ($m(e) = m(e')$), by (3.2). Clearly the dependency relation is symmetric and transitive.

Now suppose that G contains an induced subgraph $H' = (T', U')$ isomorphic to $K_{3,3}^-$ (notation $H' \simeq K_{3,3}^-$). Note that H' is isometric since G is bipartite. Moreover, it is easy to check that all edges of H' are dependent (via 4-cycles), therefore,

(3.3) the submetric μ' of m to T' is $\lambda \mathrm{d}^{H'}$ for some $\lambda > 0$.

As shown in [9], the distance function $d = \mathrm{d}^{K_{3,3}^-}$ has a nontrivial primitive extension (i.e., different from d). Moreover, the primitive extension constructed there has a submetric proportional to d. Then this submetric can again be extended in a similar way, and one can repeat such a procedure as many times as one wishes, every time obtaining a new primitive extension of the initial metric due to Statement 3.1. Such a construction is based on the following fact (a sharper version is given in [9]).

Statement 3.2. *Let μ' be a metric on a set T', and let $G'' = (V'', E'')$ be a graph with $V'' \supseteq T'$ such that: (i) for some $\alpha > 0$, $m'' = \alpha \mathrm{d}^{G''}$ is a tight extension of μ', and (ii) all edges of G'' are dependent. Then m'' is a primitive extension of μ'.*

Proof. Since m'' is tight, any two opposite nodes x, y in an (even) isometric cycle C satisfy $m''(sx) + m''(xy) + m''(yt) = m''(st)$ for some $s, t \in T'$. Hence, x, y are in some shortest T'-path in G'' whose part between x and y is a corresponding path in

C. Arguing as above, we observe that relations similar to those in (3.1)-(3.2) are valid, whence $m''(e)$ is a constant β for all $e \in E'$ (by (ii)). Considering a shortest T-path P in G'', we have $\beta = \mu'(st)/|P|$, where s, t are the ends of P. Now the fact that m'' is tight implies that any $u, v \in V''$ satisfy $m''(uv) = \beta|Q|$, where Q is a shortest u–v path in G''. Thus, m'' is determined uniquely by its m''-shortest T'-paths, whence m'' is primitive (taking into account that m'' is, obviously, positive). •

Return to consideration of the subgraph $H' = (T', U') \simeq K_{3,3}^{-}$ in G and the corresponding submetric $\mu' = \lambda \mathrm{d}^{H'}$ of m.

Statement 3.3 [9]. *For $H' = (T', U') \simeq K_{3,3}^{-}$, there exists a bipartite graph $G'' = (V'', E'')$ with $V'' \supset T'$ such that: (i) $m'' = \frac{1}{2}\mathrm{d}^{G''}$ is a primitive extension of $\mathrm{d}^{H'}$, and (ii) G'' contains an induced subgraph $K_{3,3}^{-}$.*

(The desired graph G'' is drawn in Fig. 4b where for convenience the nodes of H' are labelled by $1, \ldots, 6$ as indicated in Fig. 4a. This G'' is obtained by splitting each edge $e = ij$ of H' into two edges iz_e and $z_e j$ in series, and adding: (a) two extra nodes x and y, (b) edges xz_e for all $e = ij \in U'$ with $i, j \leq 5$, and (c) edges yz_e for all $e = ij \in U'$ with $i, j \geq 2$. One can check that $m'' = \frac{1}{2}\mathrm{d}^{G''}$ is a tight extension of $\mathrm{d}^{H'}$ (e.g., x and y belong to a shortest path of length 6 which connects nodes 1 and 4), and that all edges in G'' are dependent. So m'' is $\mathrm{d}^{H'}$-primitive, by Statement 3.2. Also G'' has an induced sugraph $K_{3,3}^{-}$; such a subgraph is drawn in bold in Fig. 4b).

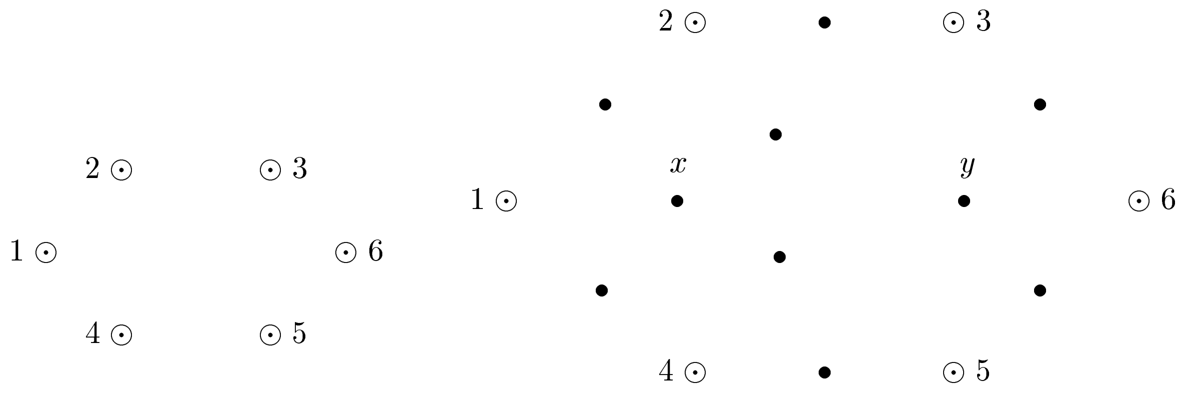

Fig. 4 (a) $H' \simeq K_{3,3}^{-}$ (b) G''

Now taking the primitive extension $\lambda m''$ of μ' (where μ', λ are as in (3.3) and m'' is as in Statement 3.3) and applying Statement 3.1 to μ, m and our μ' and $\lambda m''$, we obtain an extreme extension of μ on $V \cup V''$ coinciding with $\lambda m''$ on V''. Since G'' itself contains an induced subgraph $H'' = (T'', U'')$ isomorphic to $K_{3,3}^{-}$, we can extend the submetric $\mu'' = \lambda m''_{|T''}$ using again the construction involving a copy of the graph G'', which results in a new primitive extension of μ, and so on. Obviously, all the primitive extensions constructed this way are different, and we conclude that μ has infinitely many primitive extensions, as required.

Next we suppose that G is not hereditary modular. Since G is modular, G contains an isometric 6-cycle $C = v_0v_1 \dots v_5v_0$, by Bandelt's theorem ((ii) in Theorem 1.9). By (3.2),

$$m(v_0v_1) = m(v_3v_4) =: \alpha, \quad m(v_1v_2) = m(v_4v_5) =: \beta, \quad m(v_2v_3) = m(v_5v_0) =: \gamma.$$

Let $\widetilde{\mu}$ be the submetric of m on $\widetilde{T} = \{v_0, \dots, v_5\}$. Our goal is to find a primitive extension $\widetilde{m}$ of $\widetilde{\mu}$ such that $\widetilde{m}$ has a submetric μ' of the form $\alpha \mathrm{d}^{K^-_{3,3}}$. Then one can apply to μ' the above construction which provides infinitely many primitive extensions for the initial μ.

The desired $\widetilde{m}$ is easy to construct when $\alpha = \beta = \gamma$. More precisely, let $u_i = v_i$, $i = 0, \dots, 5$, and let Γ be the graph on eight nodes $u_0, \dots, u_5, x, y$ as drawn in Fig. 5a. One can see that d^Γ is a tight extension of d^C and that the edges of Γ are dependent. Therefore, d^Γ is a primitive extension of d^C, whence there is an extreme extension of μ to $V \cup \{x, y\}$ whose submetric on the node set of Γ is $\alpha \mathrm{d}^\Gamma$. Since Γ contains $K^-_{3,3}$ as an induced subgraph (drawn in bold in Fig. 5a), the result follows.

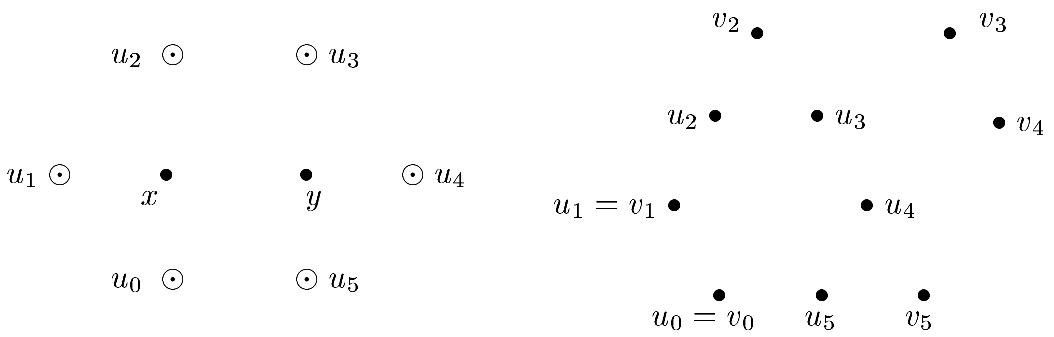

Fig. 5 (a) Γ (b)

Now suppose that the α, β, γ are not the same, $\alpha < \beta \le \gamma$ say. Let ρ be the distance function on a set W of ten points $u_0 = v_0, u_1 = v_1, v_2, \dots, v_5, u_2, \dots, u_5$, defined by

$$\begin{aligned} &\rho(v_iv_j) = m(v_iv_j) \quad \text{for } i,j = 0,\dots,5, \\ &\rho(u_iu_j) = \alpha \min\{|i-j|, 6-|i-j|\} \quad \text{for } i,j = 0,\dots,5, \\ &\rho(u_iv_j) = m(v_0v_j) - \alpha\varphi(i) \quad \text{for } i,j = 0,\dots,5, \end{aligned} \tag{3.4}$$

where indices are taken modulo 6 and $\varphi(i)$ is the distance in G between v_0 and v_i (so $0 \le \varphi(i) \le 3$). One can check that ρ is indeed a metric on W (Fig. 5b illustrates the least generating graph for ρ). Moreover, we observe from (3.4) that for each $i = 0, \dots, 5$, the ρ-length of the path $P_i = v_iu_iu_{i+1}u_{i+2}u_{i+3}v_{i+3}$ is equal to

$$\begin{aligned} &\rho(v_iu_i) + \rho(u_iu_{i+3}) + \rho(u_{i+3}v_{i+3}) \\ &\qquad = (m(v_0v_i) - \alpha\varphi(i)) + 3\alpha + (m(v_0v_{i+3}) - \alpha\varphi(i+3)) \\ &\qquad = m(v_iv_{i+3}) + 3\alpha - \alpha(\varphi(i) + \varphi(i+3)) = m(v_iv_{i+3}) \quad (= \alpha + \beta + \gamma), \end{aligned}$$

i.e., P_i is a ρ-shortest $\widetilde{T}$-path. Since each two points of W occur in some P_i, we conclude that ρ is a tight extension of $\widetilde{\mu}$.

Note that ρ is not primitive for $\widetilde{\mu}$. Nevertheless, we can use the fact that the submetric ρ' of ρ on $\{u_0,\ldots,u_5\}$ is α times the distance function of C_6. We further extend ρ by use of the metric $\alpha \mathrm{d}^{\Gamma}$, where Γ is the above-mentioned graph (that depicted in Fig. 5a). Then the resulting metric $\widetilde{\rho}$ on $W \cup \{x,y\}$ is already a primitive extension of $\widetilde{\mu}$. This follows from the observation that the collection of the above ρ-shortest paths P_i together with the shortest paths between u_i and u_{i+3} in Γ, $i=0,1,2$, determines $\widetilde{\rho}$ uniquely.

Again, μ has a primitive extension of which some submetric is proportional to $\mathrm{d}^{K^-_{3,3}}$, providing the existence of infinitely many primitive extensions for μ. This completes the proof of part (i)→(ii) in Theorem 1.2.

4. Embedding in a minimizable graph

Let the graph $G=(V,E)$ as above be a semiframe. We show that in this case there exists a frame H such that μ is a submetric of d^H. This will provide part (ii)→(i) in Theorem 1.2 and other results.

A maximal complete bipartite subgraph K in G is called a *bi-clique* if $|A|,|B| \ge 2$, where A,B are the maximal stable sets in K; we often denote K by $(A;B)$. First of all we observe the following.

Statement 4.1. *Let $K=(A;B)$ and $K'=(A';B')$ be two different bi-cliques such that the graph $K \cap K'$ is nonempty. Then $K \cap K'$ is connected and contains at most one edge. In particular, every 4-cycle of G is contained in exactly one bi-clique.*

Proof. Let for definiteness $A \cap A' \neq \emptyset$. Suppose that $A \cap A'$ contains two different nodes x,y. Since K and K' are different, w.l.o.g. we may assume that there are $u \in A$ and $v \in B'$ which are not adjacent in G. Choose two different nodes z,z' in B (existing because $|B| \ge 2$). Then the subgraph induced by $\{x,y,u,v,z,z'\}$ is $K^-_{3,3}$, contradicting the fact that G is a semiframe. Thus, $|A \cap A'| = 1$. By a similar reason, $|B \cap B'| \le 1$, and the result follows. •

A maximal set of dependent edges in G is called an *orbit*. The core of our construction of the desired minimizable graph involves orbit splitting operations that we now describe.

Note that each edge e which is not a bridge of G belongs to a cycle; moreover, e belongs to a 4-cycle and, therefore, to a bi-clique (since any minimum length cycle containing e is isometric, and G has no isometric k-cycle with $k > 4$).

Consider an orbit Q. Let $\mathcal{K}(Q)$ be the set of bi-cliques whose edge sets meet Q. Note that an edge of a 4-cycle belongs to Q if and only if the opposite edge does. This implies that all edges of each bi-clique $K = (A; B) \in \mathcal{K}(Q)$ with $|A| + |B| \geq 5$ are contained in Q. On the other hand, if $|A| = |B| = 2$, it is possible that K and Q have only two edges in common; in this case the bi-clique K is called *simple.*

The *orbit splitting operation* for Q transforms G into $G' = (V', E')$ as follows.

(4.1) (i) Split each edge $e = xy \in Q$ into two edges xz_e and yz_e in series.

(ii) If $K \in \mathcal{K}(Q)$ is simple and $K \cap Q = \{e, e'\}$, then z_e and $z_{e'}$ are connected by an edge (see Fig. 6a).

(iii) If $K = (A; B) \in \mathcal{K}(Q)$ is not simple, then a new node v_K is added and v_K is connected by an edge with z_e for each edge e of K (see Fig. 6b where $|A| = 2$ and $|B| = 3$).

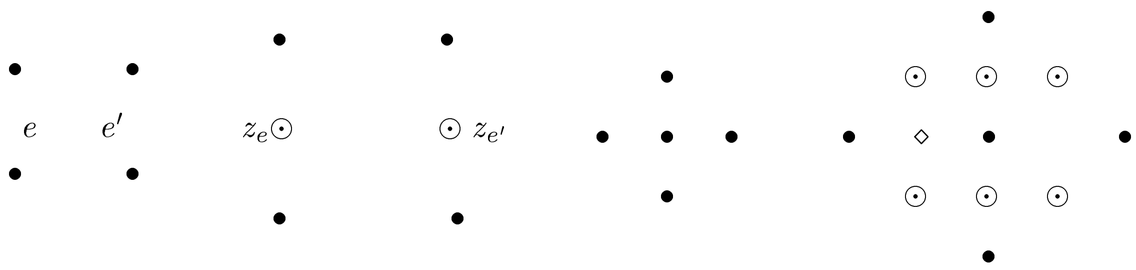

Fig. 6 (a) (b)

One can see that the resulting graph G' is bipartite as well. We call edges xz_e and yz_e in (i) *split-edges*, edges $z_e z_{e'}$ in (ii) *bridge-edges*, and edges $z_e v_K$ in (iii) *star-edges*. Let us say that an orbit Q of G is *orientable* if there exists a feasible orientation of the edges of this orbit (i.e., that matches the definition of the orientability in the Introduction). Note that Q may be orientable while the whole G is not. The graph G' has important properties exhibited in the following three lemmas.

Lemma 4.2. *G' is a semiframe.*

Lemma 4.3. *Let $Q_1, \ldots, Q_r$ be the orbits of G, and let $G' = (V', E')$ be obtained from G by the orbit splitting operation applied to Q_1. Then G' has r or $r + 1$ orbits. Moreover,*

(i) for $i = 2, \ldots, r$, Q_i induces an orbit Q'_i in G' formed by the edges of Q_i and all bridge-edges $z_e z_{e'}$ such that the 4-cycle in G containing e, e' has the other two edges in Q_i; also Q'_i is orientable if and only if Q_i is so;

(ii) if Q_1 is non-orientable, then Q_1 induces only one orbit Q'_1 in G', which is orientable

and formed by all split- and star-edges;

(iii) if Q_1 is orientable, then Q_1 induces two orbits Q'_1 and Q''_1 in G', and the set $Q'_1 \cup Q''_1$ consists of all split- and star-edges; also both Q'_1 and Q''_1 are orientable, and for each $e = xy \in Q_1$, one of the edges $xz_e, yz_{e'}$ belongs to Q'_1 while the other to Q''_1.

Lemma 4.4. *Let ρ be the length function on the edges of G defined as follows (using notation from the previous lemma):*

(4.2) (i) for $i = 2, \ldots, r$, the ρ-length of each edge in Q'_i is equal to the distance $m(e)$ of an edge $e \in Q_i$;

(ii) if Q_1 is non-orientable, then the ρ-length of each edge in Q'_i is equal to $m(e)/2$ for $e \in Q_1$;

(iii) if Q_1 is orientable and $e \in Q_1$, then choose an arbitrary number $0 < \lambda < m(e)$ and put $\rho(e') = \lambda$ for all $e' \in Q'_1$ and $\rho(e'') = m(e) - \lambda$ for all $e'' \in Q''_1$.

Then the metric m^ρ on V' generated by (G', ρ) coincides with ρ on E' and is a tight extension of μ.

These lemmas will be proved later, and now we explain how, with their help, to prove that $\Pi(\mu)$ is finite.

First we apply the orbit splitting operation consecutively to the orbits $Q_1, \ldots, Q_r$ of G (more precisely, to the images of Q_i's appeared in the current graphs). This results in a semiframe $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ and a function $\widetilde{\rho}$ on $\widetilde{E}$ such that $\widetilde{m} = m^{\widetilde{\rho}}$ is a tight extension of μ (using induction on r and repeatedly applying Lemmas 4.2-4.4). One can see that $\widetilde{G}$ can be directly constructed from G as follows (cf. (4.1)).

(4.3) (i) Split *each* edge $e \in E$ into two edges xz_e and yz_e in series.

(ii) For *each* bi-clique K of G, add a new node v_K and edges $z_e v_K$ for all edges e of K.

This fact enables us to show the following important property, which makes clear why we split the orbits of G.

Statement 4.5. *$\widetilde{G}$ is a frame.*

Proof. One can see from (4.3) that each 4-cycle $\widetilde{C}$ of $\widetilde{G}$ is of the form $xz_e v_K z_{e'} x$, where e and e' are edges in a bi-clique K of G which are incident to a node x. Therefore, we can orient each split-edge xz_e from x to z_e and each star-edge $z_e v_K$ from z_e to v_K, obtaining a feasible orientation for all 4-cycles of $\widetilde{G}$. Thus, $\widetilde{G}$ is orientable, and now the fact that $\widetilde{G}$ is a semiframe (by Lemma 4.2) implies that $\widetilde{G}$ is a frame. •

Remark. In fact, in the above method we do not need to split all orbits of G to transform it into a frame. It suffices to split only those orbits which are non-orientable.

Next, since μ can be considered up to proportionality, we may assume that all numbers $m(e)$, $e \in E$, are even integers. We also may assume that for each orientable orbit of G the number λ figured in (iii) of Lemma 4.4 is chosen to be an integer. Then the function $\widetilde{\rho}$ (as well as metric $\widetilde{m}$) is integer-valued. We now repeatedly split each orbit of $\widetilde{G}$ so as to get a frame with unit lengths of all edges.

More precisely, starting from $\overline{G} = \widetilde{G}$ and $\overline{\rho} = \widetilde{\rho}$, choose an orbit Q of the current graph $\overline{G}$ such that the current length $\overline{\rho}(e) =: \delta$ of an edge $e \in Q$ is still greater than one. Split Q with an arbitrary integer λ $(0 < \lambda < \delta)$; this transforms Q into two orbits Q' and Q'' with length λ of all edges in Q' and length $\delta - \lambda$ of all edges in Q'' (taking into account that Q is orientable, in view of Lemma 4.3). Choose an appropriate orbit in the new current graph, and so on. Eventually, we obtain a frame $\widehat{G} = (\widehat{V}, \widehat{E})$ with unit length of each edge (as before, this immediately follows by induction, in view of Lemmas 4.2-4.4). The resulting metric $\widehat{m}$ on $\widehat{V}$ is just $\mathrm{d}^{\widehat{G}}$ (by Lemma 4.4), and $\widehat{m}$ is a tight extension of μ. In particular, μ is a submetric of $\mathrm{d}^{\widehat{G}}$.

Consider a primitive extension m' of μ to a set V'. Since μ is a submetric of $\mathrm{d}^{\widehat{G}}$, there exists an extreme extension m'' of $\mathrm{d}^{\widehat{G}}$ to $\widehat{V} \cup V'$ such that m'' and m' coincide within V' (by a weakened version of Statement 3.1). Shrinking the zero distance sets for m'' makes a primitive extension d of $\mathrm{d}^{\widehat{G}}$ to a set Z. Since $\widehat{G}$ is a frame, $Z = \widehat{V}$ and $d = \mathrm{d}^{\widehat{G}}$. Furthermore, the submetric of d on (the image of) V' is exactly m'. Thus, the number of primitive extensions of μ does not exceed the number of sets V' such that $T \subseteq V' \subseteq \widehat{V}$, whence $\Pi(\mu)$ is finite. This gives part (ii)→(i) in Theorem 1.2.

The above construction and Theorem 1.2 give rise to other results mentioned in the Introduction. In particular, the finiteness of the set of submetrics of $\mathrm{d}^{\widehat{G}}$ implies part "if" in Theorem 1.1. On the other hand, part "only if" of Theorem 1.1 is provided by (i)→(ii) of Theorem 1.2 and the above construction of the frame $H = \widehat{G}$. Furthermore, observe that the tight spans of μ and $d = \mathrm{d}^{\widehat{G}}$ are the same. Indeed, since d is a tight extension of μ, it is easy to see that any tight extension of d is simultaneously a tight extension of μ. Conversely, since μ is a submetric of d, any tight extension of μ can be expanded to a tight extension of d. As mentioned in the Introduction, the tight span of the path metric of any frame is 1- or 2-dimensional; so is the tight span of $\mathrm{d}^{\widehat{G}}$. This gives Theorem 1.3 and Corollary 1.6, taking into account that the tight spans of $\mathrm{d}^{K_{3,3}^-}$ and d^{C_6} are 3-dimensional, e.g., by Dress' result (Theorem 1.7(ii)). Finally, if μ is cyclically even and $\Pi(\mu)$ is finite, then a median closure m of μ is cyclically even (by Statement 2.1). If we apply the above construction to the metric $2m$ (whose all values are even), then $2m$ is a submetric of $\mathrm{d}^{\widehat{G}}$ for the resulting frame $\widehat{G}$. This gives Corollary 1.5.

It remains to prove Lemmas 4.2-4.4. Let $Q_1, \dots, Q_r$ be the orbits of G, let $G' = (V', E')$ be formed by the orbit splitting operation applied to $Q = Q_1$. We call nodes of G' of the form z_e and v_K *split-* and *star-nodes*, respectively.

Proof of Lemma 4.3. (i) in this lemma is easy.

To see (iii), fix an orientation of the edges of Q and form Q' and Q'' as follows. If $e = xy \in Q$ is oriented as (x, y), then we orient xz_e as (x, z_e) and make it be a member of Q', and orient yz_e as (z_e, y) and make it be the member of Q''.

To assigh membership for the star-edges, consider a non-simple bi-clique $K = (A; B) \in \mathcal{K}(Q)$, and choose a 4-cycle $C = v_0v_1v_2v_3v_0$ in K. One may assume that the edges of C are oriented as drawn in Fig. 2, and let $v_0, v_2 \in A$. Then $2 = |A| \le |B|$ (for if $|A| \ge 3$, then K has no feasible orientation), and for each $x \in B$, the edges v_0x and v_2x must be oriented as (v_0, x) and (x, v_2). For each $x \in B$ and $e = v_0x$, orient z_ev_K as (z_e, v_K) and include it in Q', while for $e' = v_2x$, orient $z_{e'}v_K$ as $(v_K, z_{e'})$ and include it in Q''.

One can check that each of Q', Q'' is indeed an orbit of G' and that the orientation we constructed is feasible for both Q' and Q''.

To see (ii), orient each split-edge xz_e as (x, z_e) and each star-edge z_ev_K as (z_e, v_K). This gives a feasible orientation of the split- and star-edges in all 4-cycles of G' where such edges occur (compare with the orientation in the proof of Statement 4.5). Next, since Q is non-orientable, for any edge $e \in Q$, there exists a sequence (*orientation-reversing dual cycle*) $D = (e^0, F^1, e^1, \dots, F^q, e^q)$ consisting of edges $e^i = x^iy^i$ and 4-cycles F^i of the form $x^{i-1}y^{i-1}y^ix^ix^{i-1}$ in G such that $e = e^0$, $x^0 = y^q$ and $y^0 = x^q$. One can see that the edges $x^0z_{e^0}, x^1z_{e^1}, \dots, x^qz_{e^q}$ of G' are dependent, and now the fact that $x^qz_{e^q} = y^0z_{e^0}$ (as $x^q = y^0$) shows that all split-edges generated by the edges of D are dependent. This easily implies that all split- and star-edges in G' are dependent, and hence, they constitute a single orbit of G'. •

It is more convenient to prove the other two lemmas together because their proofs involve some common arguments.

Proof of Lemmas 4.2 and 4.4. It falls into several claims. Let $m' = m^\rho$. We call nodes of G' of the form z_e and v_K *split-* and *star-nodes*, respectively.

Claim 1. *m' is an extension of m (and therefore, m' is an extension of μ).*

Proof. Consider a V-path $P = x_0 \dots x_k$ in G'. We have to show that $\rho(P) \ge m(x_0x_k)$. Apply induction on $|P|$. One may assume that P is simple and $|P| \ge 3$ (for if $k = 1$ then $e = x_0x_1 \in E$ and $\rho(P) = m(e)$, and if $k = 2$ then $e = x_0x_2 \in E$ and $x_1 = z_e$, whence $\rho(P) = m(e)$, by the hypotheses in (ii) and (iii) of Lemma 4.4). Also one may assume that no intermediate node of P occurs in V; otherwise partition P into two V-paths and apply induction.

We show that P can be transformed into an x_0–x_k path P' in G' such that either $|P'| < |P|$ and $\rho(P') < \rho(P)$, or $|P'| = |P|$, $\rho(P') = \rho(P)$ and P' has an intermediate node in V. In both cases the result follows by induction.

Obviously, x_1 is a split-node z_e, where $e = x_0y \in E$. If x_2 is also a split-node $z_{e'}$, then the edge e' is opposite to e in a 4-cycle $C = x_0yy'x'x_0$ of G (i.e., $e' = x'y'$). Therefore, $x_0x_1x_2x'x_0$ is a 4-cycle of G', and we have $\rho(x_0x_1) = \rho(x_2x')$ and $\rho(x_0x') = \rho(x_1x_2)$. Then the path $P' = x_0x'x_2 \dots x_k$ is as required (it contains $x' \in V$ as an intermediate node).

Now suppose that x_2 is a star-node v_K, and consider the next node x_3. Clearly x_3 is a split-node $z_{e'}$, and both e, e' belong to a 4-cycle $C = x_0yy'x'x_0$ in K. Three cases are possible.

(i) $e' = x_0x'$. Then $x_0x_1x_2x_3x_0$ is a 4-cycle in G'. We have $\rho(x_0x_3) = \rho(x_1x_2)$, whence the path $P' = x_0x_3x_4 \dots x_k$ satisfies $|P'| < |P|$ and $\rho(P') < \rho(P)$, as required.

(ii) $e' = yy'$. Then $x_1x_2x_3yx_1$ is a 4-cycle in G', whence $\rho(x_1x_2) = \rho(yx_3)$ and $\rho(x_1y) = \rho(x_2x_3)$. Therefore, the path $P' = x_0x_1yx_3 \dots x_k$ satisfies $|P'| = |P|$ and $\rho(P') = \rho(P)$ and contains $y \in V$ as an intermediate node.

(iii) $e' = x'y'$. Let $e'' = x_0x'$ and $z = z_{e''}$. Then both $x_0x_1x_2zx_0$ and $zx_2x_3x'z$ are 4-cycles in G', whence $\rho(x_0x_1) = \rho(zx_2) = \rho(x'x_3)$, $\rho(x_1x_2) = \rho(x_0z)$ and $\rho(x_2x_3) = \rho(zx_1)$. Therefore, the path $P' = x_0zx'x_3 \dots x_k$ is as required. •

For a path $P = x_0 \dots x_k$ in G, let $\gamma(P)$ denote the image of P in G', i.e., $\gamma(P)$ is obtained from P by replacing each edge $e = x_{i-1}x_i$ occurring in Q by the path $x_{i-1}z_ex_i$. Note that $\rho(x_{i-1}z_e) + \rho(z_ex_i) = m(e)$. This together with Claim 1 shows that

(4.4) for each m-shortest path P in G, the path $\gamma(P)$ is ρ-shortest, and $\rho(\gamma(P)) = m(P)$.

A path P' in G' is called *regular* if $P' = \gamma(P)$ for some path P in G. From the prove of Claim 1 one can see that

(4.5) for any s–t path L' in G' with $s, t \in V$, there is a regular s–t path L such that $|L| \le |L'|$.

By the *grid* $\Gamma_{p,q}$, or the $p \times q$ *grid*, we mean the graph whose nodes correspond to the vectors (i, j) for $i = 0, 1, \dots, p$ and $j = 0, 1, \dots, q$ and edges correspond to the pairs $\{(i, j), (i', j')\}$ with $|i-i'|+|j-j'| = 1$. A part Γ of $\Gamma_{p,q}$ bounded by two shortest paths from $s = (0,0)$ to $t = (p,q)$ is called a *net* with the *origin* s and *end* t, or an *s–t net.* In other words, Γ is a subgraph of $\Gamma_{p,q}$ induced by all nodes (i,j) satisfying $a_j \le i \le b_j$ for two sequences $0 = a_0 \le a_1 \le \ldots \le a_q \le p$ and $0 \le b_0 \le b_1 \le \ldots \le b_q = p$ with $a_j \le b_j$, $j = 0, \ldots, q$. Figure 7b illustrates a net Γ for $p = 4$ and $q = 3$. The *rightmost* (resp. *leftmost*) path from s to t in Γ is denoted by R^Γ (resp. L^Γ). Sometimes a node with coordinates (i,j) in Γ is denoted by $(i,j)_\Gamma$.

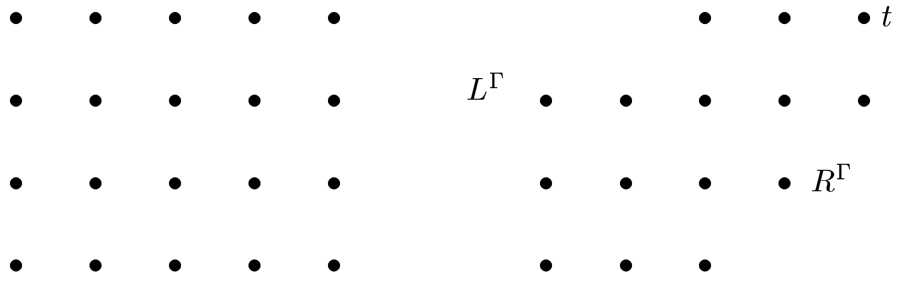

Fig. 7 (a) $\Gamma_{4,3}$ (b) Γ

In what follows we will essentially use the important property that any two shortest paths with the same ends in G can be linked by an isometric net. Such a property was proved in [9] (Claims 1 and 2 in the proof of Theorem 4.2 there) for frames; however, the proof uses only the fact that the graph in question is bipartite and contains neither isometric k-cycles with $k > 4$ nor induced subgraphs $K_{3,3}^-$; therefore, the property remains correct for an arbitrary semiframe. More precisely, the following is true:

(4.6) (i) for any $s, t \in V$ and any shortest s–t paths P and P' in Γ, there exists an s–t net Γ in G such that $R^\Gamma = P$ and $L^\Gamma = P'$;

(ii) any 2-connected net in G is isometric.

This enables us to show the following.

Claim 2. *For $s, t \in V$, let P be a shortest s–t path and P' be an s–t path in G. Then $|P \cap Q| \le |P' \cap Q|$ and $|P - Q| \le |P' - Q|$.*

Proof. First suppose that Q is also shortest. Take a net Γ as in (4.5), and let $s = (0,0)_\Gamma$ and $t = (p,q)_\Gamma$. Obviously, any two edges $e = (i,j)(i+1,j)$ and $e' = (i',j')(i'+1,j')$ with $i = i'$ are dependent; so $e \in Q$ if and only if $e' \in Q$. Similarly, any two edges $(i,j)(i,j+1)$ and $(i',j')(i',j'+1)$ with $j = j'$ are dependent. This implies that $|R^\Gamma \cap Q| = |L^\Gamma \cap Q|$, yielding the result.

Suppose that $|P'| \ge |P|$; let $P = sx_1 \ldots x_k t$ and $P' = sy_1 \ldots y_q t$. We use induction on $|P'|$. One may assume that no intermediate node of P' is in P; in particular, $x_k \ne y_q$. Take a median v for $\{s, x_k, y_q\}$. Note that by the above argument we can consider as P any shortest s–t path. So one may assume that P contains v (since v belongs to a shortest s–x_k path); let $v = x_{k'}$. Then $v \ne y_q$. If $k' = k$, then the path $L = sx_1 \ldots x_k t y_q$

is shortest (since x_kty_q is shortest, by (2.2)), and now the required inequalities for P and P' immediately follow by induction from those for L and $sy_1 \dots y_q$.

Now let $k' \neq k$. Then the nodes x_k, v, y_q are different, and the fact that any shortest x_k–x_q path has two edges (by (2.2)) implies that v is adjacent to both x_k, y_q; in particular, $k' = k-1$. Applying induction to the shortest paths $R = sx_1 \dots x_{k-1}y_q$ and $P'' = sy_1 \dots y_q$, we have $|R \cap Q| \leq |P'' \cap Q|$ and $|R - Q| \leq |P'' - Q|$. Also $x_{k-1}x_k$ is in Q if and only if y_qt is in Q, and similarly $x_{k-1}y_q$ and x_kt (regarding the 4-cycle $x_{k-1}x_kty_qx_{k-1}$). Therefore, $|P' \cap Q| - |P \cap Q| = |P'' \cap Q| - |R \cap Q|$, whence the result follows. •

This claim together with (4.4), (4.5) and Statement 2.2 shows that

(4.7) a path P in G is shortest if and only if $\gamma(P)$ is G'-shortest;

(4.8) a shortest regular path in G' is ρ-shortest.

Let d and d' stand for the metrics d^G and $\mathrm{d}^{G'}$, respectively. In order to prove the remaining statements in Lemmas 4.2 and 4.4, we need the following corollary from (4.6) (which occurred in [9] for frames):

(4.9) if $a, c, b_1, b_2 \in V$ are different nodes such that a is adjacent to both b_1, b_2 and $d(b_ic) < d(ac)$ for $i = 1, 2$, then there is a unique node $a' \in V$ which is adjacent to both b_1, b_2 and satisfies $d(ac) = d(a'c) + 2$.

Indeed, take an a–c net Γ with R^Γ containing b_1 and L^Γ containing b_2. Then $a' = (1,1)_\Gamma$ is adjacent to b_1, b_2 and satisfies $d(ac) + d(a'c) + 2$. Suppose that there is another node a'' with a similar property. Then Γ can be chosen so that R^Γ passes b_1, a' and L^Γ passes b_2, a''. Since $a' \neq a''$, w.l.o.g. one may assume that $a'' = (0,2)_\Gamma$ (and $b_1 = (1,0)_\Gamma$ and $b_2 = (0,1)_\Gamma$). By (4.6)(ii), the subnet of Γ induced by the points (i,j) for $i = 0, 1$ and $j = 0, 1, 2$ is isometric; therefore, $d(a''b_1) = 3$. So a'' cannot be adjacent to b_1; a contradiction.

For $x \in V'$, let $N(x)$ denote the set of nodes in G closest to x, i.e., $N(x)$ is $\{x\}$ if $x \in V$, $\{u, v\}$ if x is a split-node z_{uv}, and the node set of the bi-clique K if x is a star-node v_K. If $(x_0, x_1, \dots, x_k)$ is a sequence of nodes which occur in this order in a shortest path in G (resp. G'), we call this sequence G-shortest (resp. G'-shortest). The next auxiliary claim will be important later on.

Claim 3. *Let $x, y \in V'$. Then there exist $s \in N(x)$ and $t \in N(y)$ such that $d'(sx) + d'(xy) + d'(yt) = d'(st)$, i.e., (s, x, y, t) is G'-shortest.*

Proof. One may assume that $x \neq y$; otherwise the result is trivial. For $z \in V'$, let

$\varphi(z)$ denote $d'(zv)$, where $v \in N(z)$, i.e., $\varphi(z)$ is 0 if $z \in V$, 1 if z is a split-node, and 2 if z is a star-node. First of all we show that

(4.10) if $z \in V'$ and $u \in V$, and if v is an element of $N(z)$ with $d(uv)$ maximum, then there exists $v' \in N(z)$ such that (u, v', z, v) is G'-shortest.

This is obvious if $z \in V$. Suppose that z is a split-node $z_{vv'}$. Since G is bipartite, $d(uv) \neq d(uv')$; therefore, $d(uv) > d(uv')$, and (u, v', v) is G-shortest. This implies that (u, v', v) is G'-shortest (by (4.7)), whence (u, v', z, v) is G'-shortest. Now suppose that z is a star-node v_K for $K = (A; B)$. Let for definiteness $v \in A$. Choose two different nodes $p_1, p_2 \in B$. Clearly $d(up_i) < d(uv)$. By (4.9) for v, u, p_1, p_2, there is $v' \in V$ adjacent to both p_1, p_2 and satisfying $d(uv') = d(uv) - 2$. Then $v \in N(z)$ (by Statement 4.1) and (u, v', v) is G-shortest, whence (u, v', z, v) is G'-shortest.

Note that (4.10) is equivalent to saying that there exists $v' \in N(z)$ such that $d'(uv) = d'(uv') + 2\varphi(z)$, or $d(uv) = d(uv') + \varphi(z)$ (in view of (4.7)). We choose the desired $s \in N(x)$ and $t \in N(y)$ so that the distance $d(st)$ (or $d'(st)$) is maximum. By (4.10), there are $s' \in N(x)$ and $t' \in N(y)$ such that each of (s, t', y, t) and (s, x, s', t) is G'-shortest. If (s, s', t', t) is shortest, or, equivalently, $d(s't') = d(st) - \varphi(x) - \varphi(y)$, we are done. In particular, this is so if at least one of x, y is in V. So assume that $d(s't') > d(st) - \varphi(x) - \varphi(y)$ and consider possible cases.

(i) Suppose that both x, y are split-nodes. Then $x = z_{ss'}$ and $y = z_{tt'}$. Obviously, $\varphi(x) = \varphi(y) = 1$, and $d(st) - d(s't')$ is an even integer ≤ 2. So we have to consider only the case when $d(s't') = d(st) =: \alpha$. By (4.6)(i), there is an s–t net Γ in G with R^Γ containing s' and L^Γ containing t'. Observe that $d(s't') = d(st)$ is possible only if Γ is a $1 \times (\alpha - 1)$ grid in which $s' = (1, 0)_\Gamma$ and $t' = (0, \alpha - 1)_\Gamma$ (in particular, Γ is isometric, by (4.6)(ii)). Then the edges of Γ of the form $(0, j)(1, j)$ belong to the orbit Q. The split-nodes induced by these edges together with the star-nodes induced by the non-simple bi-cliques including 4-cycles from Γ generate in a natural way an x–y path P'. Moreover, the concatenation of sx, P' and yt gives an s–t path P which, obviously, has the same length as that of $\gamma(R^\Gamma)$. Hence, (s, x, y, t) is G-shortest.

(ii) Next suppose that both x, y are star-nodes, $x = v_{K=(A;B)}$ and $y = v_{K'=(A';B')}$. Assuming that $s \in A$ and $t \in A'$, we have $s' \in A$ and $t' \in A'$. Let $\alpha = d(st)$. Since $\varphi(x) + \varphi(y) = 4$, one has to consider two cases, namely, $d(s't') = \alpha$ and $\alpha - 2$.

(a) Let $d(s't') = \alpha$. Consider an s–t net Γ such that R^Γ contains s' and L^Γ contains t'. Observe that $d(s't') = d(st)$ is possible only if Γ is a $2 \times (\alpha - 2)$ grid in which s, t, s', t' have the coordinates $(0, 0), (2, \alpha - 2), (2, 0)$ and $(0, \alpha - 2)$, respectively. Note that $\alpha \geq 3$ (otherwise $K = K'$ and $x = y$). Consider the subnet Γ' of Γ induced by the points (i, j) for $i = 0, 1, 2$ and $j = 0, 1$, and take a node $q \in B$ different from $(1, 0)_\Gamma$ (q exists since $|B| \geq 2$). Since Γ' is isometric (by (4.6)(ii)) and G is bipartite, q is different from all nodes of Γ'. Let H be the subgraph of G induced by the nodes of Γ' and q. If q is adjacent to $(1,1)_\Gamma$, then H contains $K_{3,3}^-$, and if not, then H contains an isometric 6-cycle. Thus, the given case is impossible.

(b) Now let $d(s't') = \alpha - 2$. Choose $p \in B$ and $q \in B'$ with $d(pq)$ maximum; clearly $d(pq) \in \{\alpha, \alpha - 2\}$. If $d(pq) = \alpha$, take an s–t net Γ in G with R^Γ containing p, s' and L^Γ containing q, t'. Since $d(st) = d(pq)$, Γ is a $1 \times (\alpha - 1)$ grid in which p, s', t', q have the coordinates $(1,0), (1,1), (0, \alpha - 2)$ and $(0, \alpha - 1)$, respectively. This implies that (s, x, y, t) is G'-shortest (by the argument similar to that in (i)). Now let $d(pq) = \alpha - 2$. Then (s, p, q, t) is G-shortest. Choose $p' \in B - \{p\}$ and $q' \in B' - \{q\}$. We have $d(p'q) = d(pq) < d(sq)$ (by the choice of p, q); hence, there is $s'' \in A$ such that $\alpha(s''q) = \alpha - 3$. Similarly, there is $t'' \in A'$ such that $\alpha(t''p) = \alpha - 3$. Moreover, we can consider s'', t'' instead of s', t'; so we may assume that $d(s''t'') = \alpha - 2$. Then a p–q net Γ in G with R^Γ containing s'' and L^Γ containing t'' is a $1 \times (\alpha - 3)$ grid. One can see that none of s, t, p', q' occurs in Γ. Therefore, adding to Γ these nodes and the edges $sp, sp', p's'', tq, tq', q't''$ makes a $1 \times (\alpha - 1)$ grid Γ'. Since Γ' is isometric (by (4.6)(ii)), $d(p'q') = \alpha$, contrary to the choice of p, q.

The remaining case when one of x, y is a split-node and the other is a star-node combines arguments from (i) and (ii) and is left to the reader as an exercise. •

This claim shows that any two nodes $x, y \in V'$ are contained in a G'-shortest s–t path L for some $s \in N(x)$ and $t \in N(y)$. Moreover, starting from an arbitrary L with such a property, one can transform it, step by step, into a regular G'-shortest s–t path L' (not necessarily including x and y) by use of elementary transformations described in the proof of Claim 1. We have seen that each elementary transformation does not change the ρ-length of a path (when its G'-length preserves). This together with the fact that the resulting path L' is ρ-shortest (by (4.8)) shows that the initial path L is ρ-shortest as well. Thus, $m^\rho(sx) + m^\rho(xy) + m^\rho(yt) = m^\rho(st) = m(st)$ (in view of Claim 1), i.e., m^ρ is a tight extension of m and, therefore, m^ρ is a tight extension of μ. Also, considering L for the case when xy is an edge of G', we conclude that $\rho(xy) = m^\rho(xy)$. So Lemma 4.4 is proven.

We now use Claim 3 to show that G' is a semiframe.

Claim 4. *G' is modular.*

Proof. By (iii) in Theorem 1.9, it suffices to prove the existence of a median for a triple $\{u_1, u_2, y\}$ in V' such that $d'(u_1u_2) = 2$ and $d'(u_1y) = d'(u_2y) \geq 2$. Let $x \in V'$ be adjacent to both u_1, u_2. If $d'(xy) = d'(u_1) - 1$, we are done. So assume that $d'(xy) = d'(u_iy) + 1$, $i = 1, 2$.

Choose s,t as in Claim 3 for our x,y. Then

$$\text{(4.11)} \qquad \text{for } i=1,2, \quad (s,x,u_i,y,t) \text{ is } G'\text{-shortest.}$$

Let L_i be a G'-shortest s–t path containing x,u_i,y. We examine possible cases for x.

(i) Suppose that x is a star-node v_K for $K=(A;B)$. Then $u_1=z_e$ and $u_2=z_{e'}$ for some edges $e=p_1q_1$ and $e'=p_2q_2$ in K. Let for definiteness $s,q_1,q_2\in A$ (and $p_1,p_2\in B$). Since $d'(su_i)>1$ (by (4.11)), $q_i\ne s$. Moreover, the s–t path L'_i obtained from L_i by replacing its part from s to u_i by $sz_{sp_i}p_iu_i$ is again G'-shortest. This implies $d'(q_it)<d'(p_it)<d'(st)$, whence $d(q_it)<d(p_it)<d(st)$ (by (4.7)). Therefore,

$$\text{(4.12)} \qquad \text{for } i=1,2, \quad (s,p_i,q_i,t) \text{ is } G\text{-shortest.}$$

By (4.9) for s,t,p_1,p_2, there is a unique node in G adjacent to both p_1,p_2 and closer to t than p_i. Hence, $q_1=q_2=:q$ (since any p_i and q_j are adjacent). If $y\in V$ (i.e., if $y=t$), then q is a median for u_1,u_2,y in G' (in view of (4.12) and (4.7)). We show that q is a median also in two remaining cases for y.

Let y be a split-node $z_{tt'}$. Considering the above G'-shortest path L'_i and using (4.7), we observe that $d(p_it)<d(p_it')$ is impossible. Therefore, (s,p_i,t',t) is G-shortest (in view of (4.12)). By (4.9), there is a node $q'\in V$ adjacent to p_1,p_2 and satisfying $d(q't')<d(p_it')$. If $q'\ne q$, we apply (4.9) to p_1,t,q,q' (using $d(tq)+d(qp_1)=d(tq')+d(q'p_1)$) to obtain a node q'' adjacent to q,q' and satisfying $d(q''t)<d(qt)$. One can see that the subgraph of G induced by $\{s,p_1,p_2,q,q',q''\}$ is $K^-_{3,3}$; a contradiction. Thus, $q=q'$, and now the fact that (p_i,q',t',t) is G-shortest implies that (u_i,q,y) is G'-shortest, whence q is a median for $\{u_1,u_2,y\}$.

Now let y be a star-node $v_{K'}$ for $K'=(A';B')$. Then the last four nodes of the above G'-shortest path L'_i are $z'=z_{t'g},v_{K'},z_{ht},t$ (in this order). Assuming $t\in A'$ and $g,h\in B'$, we have $t'\in A-\{t\}$. Moreover, $d'(z't')=1<3=d'(z't)$ implies $d(p_it')<d(p_it)$, whence $d(p_it)=d(p_it')+2=d(p_it')+d(t't)$. Therefore, (s,p_i,t',t) is G'-shortest, and now we argue as in the previous case.

(ii) Next suppose that x is a split-node z_e, where $e=ss'$. Consider five possible cases, depending on the types of u_1,u_2 (symmetric cases are omitted).

(a) Let u_1 be a split-node z_{pq} and $u_2\in V$; then $u_2=s'$. Assuming w.l.o.g. that s and p are adjacent in G, replace in L_1 the node x by p, forming the G'-shortest s–t path L'_1. Then $d(pt),d(s't)<d(st)$, so each of (s,p,q',t) and (s,s',q',t) is G-shortest for some $q'\in V$ adjacent to both p,s' (by (4.9)). Moreover, since (s,p,z_{pq},t) is G'-shortest, we have $d(pt)>d(qt)$. This implies that $q=q'$ (by (4.9) and the fact that q and s' are adjacent). Therefore, q is a median for (u_1,u_2,t), and now, arguing as in (i), we can show that q is a median for (u_1,u_2,y) too.

(b) Let u_i be a split-node $z_{p_i q_i}$ for $i = 1, 2$, and assume w.l.o.g. that both p_1, p_2 are adjacent to s. For $i = 1, 2$, (s, p_i, u_i, t) is G'-shortest, whence (s, p_i, q_i, t) is G-shortest. Take $q \in V$ adjacent to both p_1, p_2 and giving $d(qt) < d(p_i t)$. Note that all nodes in $Z = \{s, s', p_1, q_1, p_2, q_2\}$ are different (for if $q_1 = q_2$, then the above nodes are contained in the same bi-clique, so x and u_i cannot be adjacent). One can see that if q coincides with some of these nodes (it can be only q_1 or q_2), then the induced subgraph on Z is $K^-_{3,3}$, while if q is not in Z, then either the 6-cycle $qp_1q_1s'q_2p_2q$ is isometric (when $qs' \notin E$) or the induced subgraph on $\{q, s, s', p_1, p_2, q_1\}$ is $K^-_{3,3}$ (otherwise). Thus, this case is impossible.

(c) Let u_1 be a split-node z_{pq} and u_2 a star-node v_K. Assuming that $sp \in E$ and sp' is an edge in K with $p' \neq s'$, and considering corresponding G'-shortest s–t paths, we observe that $d(pt), d(p't) < d(st)$ and conclude that this case is impossible by a similar reason as in (b).

(d) In the case when both u_1, u_2 are star-nodes, we come to a similar contradiction.

(e) Let u_1 be a star-node v_K, and $u_2 \in V$; then $u_2 = s'$. This case is analogous to (a). More precisely, for an edge sp in K with $p \neq s'$, (s, z_{sp}, u_1, t) is G'-shortest, implying $d(pt) < d(st)$. Also $d(s't) < d(st)$. Therefore, $\{p, s', t\}$ has a median q (in G); clearly q is in K. Arguing as in (i), observe that q is a median for $\{p, s, y\}$ in G'. This implies that $z_{s'q}$ is a median for $\{u_1, u_2, y\}$.

(iii) Finally, suppose that $x \in V$, i.e., $x = s$. Up to symmetry, three cases are possible. (a) If $u_1, u_2 \in V$, take a median q for $\{u_1, u_2, t\}$ in G. Then q is a median for $\{u_1, u_2, y\}$ in G' (by the argument in (i)). (b) Let both u_1, u_2 be split-nodes; then $u_1 = z_{sp}$ and $u_2 = x_{sp'}$ for some $p, p' \in V$. We have $d(pt), d(p't) < d(st)$. A median q for $\{p, p', t\}$ in G is a median for $\{p, p', y\}$ in G'. Then the star-node v_K of the bi-clique K including the 4-cycle $spqp's$ is a median for $\{u_1, u_2, y\}$. (c) Let u_1 be a split-node z_{sp} and $u_2 \in V$. Then a median q for $\{p, u_2, t\}$ is a median for $\{p, u_2, y\}$, and the desired median for $\{u_1, u_2, y\}$ is the split-node $z' = z_{u_2 q}$ (observing that u_1 and z' are connected by a bridge-edge).

Thus G' is modular. •

Next two claims finish the proof that G' is a semiframe. Let $G(x_0, \ldots, x_k)$ denote the subgraph of G induced by (not necessarily distinct) nodes $x_0, \ldots, x_k \in V$, and similarly for G'.

Claim 5. *G' has no induced subgraph $K^-_{3,3}$.*

Proof. We use the fact that $K^-_{3,3}$ is non-orientable and all its edges are dependent. Suppose that G' has an induced subgraph $F = (X, W) \simeq K^-_{3,3}$. Then all edges of F belong to the same orbit $\widetilde{Q}$ in G'. By Lemma 4.3, each orbit of G' induced by Q_1 (i.e.,

Q'_1 or Q''_1) is orientable. Therefore, $\widetilde{Q} = Q'_i$ for some $2 \le i \le r$.

We know that no induced subgraph of G is homeomorphic to $K^-_{3,3}$, so at least one edge of F is not in G. Moreover, F contains no split- or star-edges (since such edges belong to Q'_1 or Q''_1). Therefore, each node of F is a split-node and each edge of F is a bridge-edge. Let the nodes of F be numbered as shown in Fig. 1b, and let a node i correspond to an edge u_iv_i in G so that for each $ij \in W$, u_iu_j and v_iv_j are edges of G. Observe that all $u_1, \ldots, u_6$ are different (e.g., $u_1 = u_6$ would imply that G is not bipartite, and $u_2 = u_4$, say, would imply that the subgraph $G(u_1, u_2, v_1, v_2, v_4)$ is $K_{2,3}$ and, therefore, its edges belong to Q_1). So $\Gamma = G(u_1, \ldots, u_6)$ includes $K^-_{3,3}$. Since Γ is bipartite and must be different from $K^-_{3,3}$, Γ is $K_{3,3}$. Similarly, $\Gamma' = G(v_1, \ldots, v_k)$ is $K_{3,3}$. But then the nodes 1 and 6 of F should be connected by a bridge-edge; a contradiction. •

Claim 6. *G' is hereditary modular.*

Proof. Suppose this is not so. Since G' is modular (by Claim 4), G' contains an isometric 6-cycle C; moreover, C is contained in (the skeleton graph of) a cube H in G', by (ii) in Theorem 1.9. Let for definiteness H is formed by nodes $x_0, \ldots, x_3, y_0, \ldots, y_3$ and edges x_ix_{i+1}, y_iy_{i+1} and x_iy_i, $i = 0, 1, 2, 3$ (taking indices modulo 4). Note that H is an induced subgraph of G' (for if $C = x_0x_1y_1y_2y_3x_3x_0$ say and y_0 is adjacent to x_2, then $G(x_0, x_1, y_1, y_2, x_2, y_0)$ is $K^-_{3,3}$, contrary to Claim 5). Suppose that some node of H, y_0 say, is a star-node v_K. Then x_0, y_1, y_3 are split nodes corresponding to some edges of K. This easily implies that x_1, y_2, x_3 are nodes of K and that each two of these are connected by an edge, which is impossible.

So, all nodes of H are only split-nodes or nodes of G. Suppose that some edge of H, x_0y_0 say, is a split-edge; let for definiteness $x_0 \in V$ and $y_0 = e_{x_0v_0}$. Then $x_i \in V$ and y_i is a split-node $e_{x_iv_i}$ for each $i = 0, 1, 2, 3$. We observe that the subgraph $\Gamma = G(x_0, \ldots, x_3, v_0, \ldots, v_3)$ includes a cube. Indeed, obviously, $v_iv_{i+1} \in E$ and $v_i \ne x_{i+2}$ (otherwise Γ is not bipartite). Also $v_0 \ne v_2$ (otherwise $G(x_0, x_1, x_2, x_3, v_0)$ is $K_{2,3}$, whence the edge x_0x_1 must be split). Similarly, $v_1 \ne v_3$. So all $x_0, \ldots, x_3, v_0, \ldots, v_3$ are different, and Γ includes a cube Γ'. Case $\Gamma' = \Gamma$ is impossible. And if $\Gamma \ne \Gamma'$, then the edges of $\Gamma - \Gamma'$ are only some of those which connect opposite nodes of Γ'. But if there are $\eta = 3$ such edges, then Γ contains an induced $K^-_{3,3}$, while if $\eta = 4$, then Γ' turns into $K_{4,4}$, which is also impossible.

When all the nodes $x_0, \ldots, x_3, y_0, \ldots, y_3$ are in V, these induce a subgraph of G including the cube H, and we also easily come to a contradiction. Finally, if all nodes of H are split-nodes, then all its edges are bridge-edges. Let for definiteness $x_i = z_{u_iv_i}$, and u_i is adjacent to u_{i+1} in G, $i = 0, 1, 2, 3$. Considering $\Gamma = G(u_0, \ldots, u_3, v_0, \ldots, v_3)$ and arguing as above (for $G(x_0, \ldots, x_3, v_0, \ldots, v_3)$) we again observe that Γ includes a cube Γ' and that each edge of $\Gamma - \Gamma'$ connects opposite nodes of Γ', coming to a contradiction. •

This completes the proof of Lemmas 4.2 and 4.4 (yielding (ii)→(i) in Theorem 1.2). • •

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