On a conjecture of Kierstead *

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Abstract

The aim of this note is to disprove a conjecture of H. Kierstead [5].

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1 Introduction

For a natural n let $[n] = \{1, 2, \ldots, n\}$ and $P = (X, \leq)$ be a finite partially ordered set. A linear extension of P is an injection $\sigma : X \to [|X|]$ such that $x \leq y$ implies $\sigma(x) \leq \sigma(y)$. The dimension dim P of P is the least t such that there is a collection Σ of t linear extensions such that not $x \leq y$ implies $\sigma(y) < \sigma(x)$, for some $\sigma \in \Sigma$.

Let B_n denote the Boolean lattice obtained by ordering the power set of $[n]$ by inclusion. Let $B_n(j, k)$ be the restriction of B_n to subsets of size j and k, i.e. $B_n(j, k)$ consists of the j's and k's level of B_n and the respective ordering by inclusion. Let $\dim(j, k; n) = \dim B_n(j, k)$. The important special case which appeared in many papers is $\dim(1, k; n)$. To understand better why this is important we need some definitions.

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The comparability graph G of a partially ordered set P is defined to be $G = (X, E)$, where $E = \{xy : x, y \in X and\ x < y \text{ or } y < x\}$. For each positive integer Δ, let $g(\Delta)$ denote the maximum value of the dimension of an ordered set whose comparability graph has maximum degree Δ. The main result in [3] is the following upper bound of $g(\Delta)$:

For all Δ,

$$g(\Delta) = O(\Delta \log^2 \Delta). \tag{1}$$

To prove this result Fűredi and Kahn proved that if P is an ordered set whose comparability graph has maximum degree Δ, then

$$\dim P = O(\frac{\Delta}{\log \Delta} \dim(1, \log \Delta; \Delta \log \Delta)).$$

Erdős, Kierstead and Trotter [2] used probabilistic methods to show that

$$g(\Delta) = \Omega((\Delta \log \Delta).$$

Thus there has been considerable interest in the correct value of $\dim(1, \log n; n)$. In [6] it was proved that

$$\dim(1, \log n; n) \leq \log^3 n.$$

Kierstead [5] proved that

$$\dim(1, \log n; n) = \Omega(\log^3 n / \log \log n).$$

To prove this he used well known combinatorial objects, so-called (n, k, l)-packings.

A family F of k-subsets of $[n]$ is a (n, k, l)-packing iff the intersection of any two distinct elements has cardinality less than l. Let $m(n, k, l)$ be the greatest integer M such that there exists a (n, k, l)-packing of cardinality M.

2 A conjecture

In [5] the following conjecture was stated.

Conjecture. There exists a constant $c > 0$ such that with the following notations