ON ISOMETRIES OF THE HYPERBOLIC FIBONACCI MANIFOLDS *

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Abstract

The closed orientable hyperbolic three-dimensional manifolds uniformized by the Fibonacci groups were constructed by H.Helling, A.Kim and J.Mennicke. These manifolds can be obtained as the cyclic branched coverings of the figure-eight knot. In the present paper the isometry groups of these manifolds are described. The quotient orbifolds arising under the isometries actions are discussed.
Keywords: isometry group, hyperbolic 3-manifold, cyclic covering.

1. Introduction

In the present paper we study the isometry groups of the three-dimensional orientable closed hyperbolic manifolds uniformized by the Fibonacci groups. Recall, that *the Fibonacci groups* $F(2,m)$, $m \geq 3$, were introduced by Conway [3] and are defined by the following presentation:

$$F(2,m) = \langle x_1, \ldots, x_m \mid x_i\, x_{i+1} = x_{i+2} \quad i = 1, \ldots, m \rangle,$$

where all suffices are taken by mod m.

The review of some results about finite and arithmetic properties of the Fibonacci groups and of their generalizations can be found in [24]. The using of topological and geometrical methods for investigation of the Fibonacci group was started in the paper [9] of H. Helling, A. Kim and J. Mennicke. They constructed a family of three-dimensional manifolds M_n, $n \geq 2$, such that $\pi_1(M_n) \cong F(2,2n)$. Moreover, it was shown in [9] that manifolds M_n have geometric structures of the constant curvature. More exactly, the group

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$F(2,4) \cong \mathbb{Z}_5$ acts by isometries on the spherical space $\mathbb{S}^3$ and the manifold M_2 is the lens space $L(5,2)$. The group $F(2,6)$ is isomorphic to the Euclidean crystallographic group acting in the space $\mathbb{E}^3$ and the manifold M_3 is the Euclidean Hanzsche-Wendt manifold [29]. For $n \geq 4$ the group $F(2,2n)$ is isomorphic to a discrete cocompact subgroup of the group $\mathrm{PSL}(2,\mathbb{C})$ acting in the Lobachevsky space $\mathbb{H}^3$ without fixed points. Manifolds M_n, $n \geq 4$, are *hyperbolic*, i.e. have a constant negative curvature. Manifolds M_n introduced in [9] are referred to *Fibonacci manifolds*.

Recall some geometric and topological properties of the manifolds M_n. In [26] the exact formulae for volumes of the hyperbolic Fibonacci manifolds were obtained in terms of the Lobachevsky function. It was shown in [11] that for $n \geq 2$ the manifold M_n is the n-fold cyclic covering of the three-dimensional sphere branched over the *figure-eight knot*. By [27] for $n \geq 2$ the manifold M_n is the two-fold covering of the three-dimensional sphere, branched over the closed three-strings braid $(\sigma_1\sigma_2^{-1})^n$. In particular, M_2 is the two-fold covering branched over the figure-eight knot and M_3 is the two-fold covering branched over the three components link *Borromean rings*. In [14] the Fibonacci manifolds were obtained by $1/0$ - Dehn surgery on the manifold $\mathcal{M}(\phi)$ that is the punctured torus bundle. In this case the homeomorphism ϕ is induced by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2$.

In the present paper we develop the geometric approach to the studding of the Fibonacci groups. We investigate the isometry groups of the hyperbolic Fibonacci manifolds. In virtue of the rigidity theorem [1], the isometry group $Isom(M_n)$ of a hyperbolic manifold M_n is isomorphic to the group $Out(F(2,2n))$ of outer automorphisms of its fundamental group

$$F(2,2n) = \langle x_1, \ldots, x_{2n} \mid x_i\, x_{i+1} = x_{i+2} \quad i = 1, \ldots, 2n \rangle.$$

The obvious automorphism $x_i \longrightarrow x_{i+1}$, $i = 1, \ldots, 2n$, generates a cyclic subgroup $\mathbb{Z}_{2n}$ of the isometry group $Isom(M_n)$. In [13] N. Kuiper asked: is the group $Isom(M_n)$ isomorphic to the cyclic group of order $2n$? The negative answer follows from [27] where it is shown that the manifold M_n admits an involution which does not belong to the group $\mathbb{Z}_{2n}$. Moreover [14], all foliation preserving homeomorphisms of $\mathcal{M}(\phi)$ induce homeomorphisms of a Fibonacci manifold. The group of foliation preserving homeomorphisms of the manifold $\mathcal{M}(\phi)$ can be found in [16] and it is of order $8n$.