Actions of parabolic subgroups of $GL(V)$ on certain unipotent subgroups and quasi-hereditary algebras

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Abstract

A parabolic subgroup P of the general linear group over an infinite field acts by conjugation on its unipotent radical P_u and the members $P_u^{(l)}$ of the descending central series of P_u. Let $\ell(P_u)$ denote the nilpotency class of P_u. We show that P acts with a finite number of orbits on $P_u^{(l)}$ precisely when $\ell(P_u) \leq 4$ for $l = 0$, or $\ell(P_u) \leq 5 + 2l$ for $l \geq 1$. This result is obtained by interpreting the orbits of P on $P_u^{(l)}$ as isomorphism classes of good modules over certain quasi-hereditary algebras.

1 Introduction

(1.1) Let G be a reductive algebraic group defined over an algebraically closed field k. Let P be a parabolic subgroup of G. It acts by conjugation on its unipotent radical P_u. By a theorem of R.W. Richardson, P admits a dense orbit on P_u, see [Ri]. Recently, L.Hille and G.Röhrle classified in [HR1] and [HR2] those parabolic subgroups of classical groups which act with a finite number of orbits on their unipotent radical.

The strategy to obtain this classification is the following: At first, the cases where G is of type B, C or D are reduced to those of type A by interpreting the classical groups $SO_m(k)$ and $Sp_m(k)$ as fixed point subgroups of $SL_m(k)$ under a suitable automorphism. Secondly, if G is a simple algebraic group of type A, the orbits of P in P_u are identified with certain isomorphism classes of modules over some finite-dimensional algebra. The purpose of this paper is to extend the second step, the translation from orbits of a parabolic subgroup of a general linear group to modules over algebras in more general situations.

(1.2) Let V be a finite-dimensional k-vector space endowed with a flag of subspaces $F : \{0\} = V_0 \subseteq \cdots \subseteq V_t = V$ of length t. The stabilizer P of F in the general linear group of V is a parabolic subgroup of $GL(V)$. The members of the descending central series of P_u are defined by $P_u^{(l)} := [P_u, P_u^{(l-1)}]$, where $P_u^{(0)} := P_u$ and $l \in \mathbb{N}$. If the flag F is *proper*, i.e. $V_{i-1} \neq V_i$ for $i = 1, \dots, t$, then the nilpotency class of P_u is $\ell(P_u) = t - 1$.
Note that instead of the action of P on $P_u^{(l)}$ we may as well consider the adjoint action of P on the corresponding Lie algebras $\mathfrak{p}_u^{(l)}$. We identify $\mathfrak{p}_u^{(l)}$ with the space of all endomorphisms f of V such that $f(V_i) \subseteq V_{i-l-1}$ for $i = l+1, \dots, t$.

The main result of this paper is the following:

Theorem. *Let k be an infinite field and let $P = Stab(F)$ be a parabolic subgroup of a general linear group, realized as stabilizer of a proper flag F of length t. Then the number of orbits of P on $\mathfrak{p}_u^{(l)}$ is finite precisely when $t \leq 5$ for $l = 0$, or $t \leq 6 + 2l$ for $l \geq 1$.*

Note that the result for $l = 0$ is shown in [HR1] and [HR2].

(1.3) The proof of this Theorem generalizes the methods of [HR1]. In a first step we consider simultaneously the pairs (F, f) where F is a flag of fixed length t with associated parabolic subgroup $P = \mathrm{Stab}(F)$ and $f \in \mathfrak{p}_u^{(l)}$ for some fixed $l \in \mathbb{N}$. These pairs can be naturally viewed as objects of a category $\mathcal{F}(t, l)$ whose isoclasses for a fixed flag F correspond bijectively to the orbits of P on $\mathfrak{p}_u^{(l)}$. The advantage of this approach is that $\mathcal{F}(t, l)$ is an additive category, hence we are reduced to the question: When does $\mathcal{F}(t, l)$ have finitely many indecomposable objects?

The second step follows an idea of Gabriel: We show that $\mathcal{F}(t, l)$ is equivalent to the category of Δ-good modules over a quasi-hereditary algebra $\mathcal{A}(t, l)$. In case $l = 0$, the algebra $\mathcal{A}(t, 0)$ is the Auslander-algebra of the truncated polynomial ring $k[X]/X^t$ (see [HR1]).

The last step extends the techniques developed in [DR] for the case $l = 0$: The category of good $\mathcal{A}(t, l)$-modules is shown to be almost equivalent to the category of modules over some algebra $\mathcal{B}(t, l)$. This allows us to apply the classification results for finite-dimensional algebras, which in turn yields the main result.

(1.4) In the last section of this paper, we discuss two generalizations of the main theorem. At first, we determine which of the representation-infinite algebras $\mathcal{B}(t, l)$ are tame and which are wild. Secondly, we study the P-orbits on arbitrary ideals of $\mathfrak{p}$ in $\mathfrak{p}_u$. We refer to a computer program which allows us to solve the finiteness question for these ideals and their quotients when the nilpotency degree of P_u is sufficiently small. In the finite cases, the program computes representatives of all orbits. In fact, these representatives are independent of the field k, even if k is a finite field.

We refer to [R2] and [DR] for an introduction to quasi-hereditary algebras. Concerning the theory and the classification results for finite-dimensional algebras we refer to [HV],[R1],[ARS] and [GR].

2 From orbits of parabolic subgroups of the general linear group to Krull-Schmidt categories

Let k be a field. All vector spaces are defined over k.

The first step to prove the Theorem is to collect all orbits of all P in $\mathfrak{p}_u^{(l)}$ for fixed $l, t \in \mathbb{N}$ into one additive category $\mathcal{F}(t, l)$. The objects of $\mathcal{F}(t, l)$ are the pairs (F, f), where $F : \{0\} = V_0 \subseteq \cdots \subseteq V_t = V$ is a flag of finite-dimensional k-vector spaces of length t and f is an element of $\mathfrak{p}_u^{(l)}$. A morphism $\phi : (F, f) \to (F', f')$ is a linear map $\phi : V \to V'$ which respects the flags, that is $\phi(V_i) \subseteq V_i'$, and commutes with the endomorphisms: $\phi f = f' \phi$.