A NOTE ON NORMAL ABELIAN SUBGROUPS IN PARABOLIC GROUPS

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Abstract. Let G be a reductive algebraic group, P a parabolic subgroup of G with unipotent radical P_u, and A a closed connected unipotent subgroup of P_u which is normalized by P. We show that P acts on A with finitely many orbits provided A is abelian. This generalizes a well-known finiteness result, namely the case when A is central in P_u. Also, we obtain an analogous result for the adjoint action of P on invariant linear subspaces which are abelian as subalgebras of the Lie algebra of P_u under some mild characteristic restrictions.

1. Introduction

Throughout, G denotes a (connected) reductive algebraic group defined over an algebraically closed field k of characteristic $p \geq 0$ and P is a parabolic subgroup of G with unipotent radical P_u. The aim of this note is the following result.

Theorem 1.1. *Let G be a reductive algebraic group, P a parabolic subgroup of G, and A a closed connected normal subgroup of P in P_u. If A is abelian, then P has finitely many orbits on A.*

The particular case when A is in the center of P_u is well-known. Then the action factors through a Levi factor of P. Here, the finiteness is due to Vinberg [35, §2] in characteristic 0 (see also Kac [14]) and follows in general from work of Richardson [24, §3]. For a detailed account of the orbit structure in this situation, see [20] and [25, §2, §5]. Although, many further instances of Theorem 1.1 are surely known, as stated here, it does not seem to appear in the literature.

Observe that for abelian P-invariant subfactors in P_u, the analogous statement of the theorem is false in general. Indeed, this fact is the basis for constructing entire families of parabolic subgroups which admit an infinite number of orbits on the unipotent radical, or its Lie algebra, e.g., see [21], [22], [26], and [27]. Examples in this context also show that a parabolic subgroup may already have an infinite number of orbits on a normal subgroup of nilpotency class two, cf. [9], [15].

There are several recent articles related to this subject. For instance, all parabolic subgroups P of classical algebraic groups with a finite number of orbits on P_u are determined in [11] and [12]. Similar results for exceptional groups are obtained in [13]. In [8] all such P in $GL_n(k)$ are classified with a finite number of orbits on a given term of the lower central series of P_u. More generally, in [21] and [28] the number of parameters on which the action of P on P_u depends upon, i.e. the modality of this action, is investigated.

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Thanks to Brundan's recent extension of the classification of spherical Levi subgroups of reductive groups to arbitrary characteristic [7, §4] (see Section 3), we immediately obtain a proof of Theorem 1.1 for all cases where A is contained in the unipotent radical of a parabolic subgroup whose Levi factor is spherical in G. In particular, this includes all possibilities when G is classical (see Section 4). In the other instances where we cannot appeal to spherical Levi subgroups directly, a construction from [1] allows us to apply these results partially.

It would be very desirable to have a uniform proof of Theorem 1.1 free of case analysis, even for the expense of some characteristic restrictions.

2. Notation and Preliminaries

We denote the Lie algebra of G by $\operatorname{Lie} G$ or $\mathfrak{g}$ and the identity element of G by e; likewise for subgroups of G. Let T be a fixed maximal torus in G and $\Psi = \Psi(G)$ the set of roots of G with respect to T. Fix a Borel subgroup B of G containing T and let $\Sigma = \{\sigma_1, \sigma_2, \dots\}$ be the set of simple roots of Ψ defined by B such that the positive integral span of Σ in Ψ is $\Psi(B)$. The highest (long) root in Ψ is denoted by ϱ and, in the presence of two root lengths, the highest short one by ϱ_s. For a root β of G we denote by U_β the corresponding one-parameter unipotent subgroup of G normalized by T, and the root subspace $\operatorname{Lie} U_\beta$ by $\mathfrak{u}_\beta$. The members of U_β are called *root elements*.

Suppose that G is simple (over its center). A prime is said to be *bad* for G if it divides the coefficient of a simple root in ϱ, else it is called *good* for G, see [31, §I,4]. Furthermore, we say that a prime is *very bad* for G if it divides a structure constant of the Chevalley commutator relations for G. Thus, if $\operatorname{char} k$ is very bad for G, there are degeneracies in these relations. This only occurs if $p = 2$ and G is of type B_r, C_r, F_4, or G_2, or $p = 3$ and G is of type G_2. The same notions apply to reductive groups by means of simple components [33, 3.6].

We may assume that each parabolic subgroup P of G considered contains B.

Let N be a closed connected normal subgroup of P in P_u. Since N is normalized by $T \subset P$, the root spaces of $\mathfrak{n}$ relative to T are also root spaces of $\mathfrak{g}$ relative to T, and the set of roots of N with respect to T, $\Psi(N)$, is a subset of Ψ. Suppose that $\Psi(N)$ is closed under addition in Ψ. Note that this is automatically satisfied provided $\operatorname{char} k$ is not very bad for G. Then $\mathfrak{n} = \oplus \mathfrak{u}_\beta$ $(\beta \in \Psi(N))$ and consequently, $N = \prod U_\beta$, where the product is taken in some fixed order over $\Psi(N)$. The *support* of an element x in N, denoted by $\operatorname{supp} x$, consists of all roots β for which the projection $N \to U_\beta$ is nontrivial when evaluated on x. By the *shape of a root* $\beta = \sum n_\sigma(\beta)\sigma$ $(\sigma \in \Sigma)$ *relative to* P, we mean the sub-sum over the elements of $\Sigma(P_u) = \Psi(P_u) \cap \Sigma$, see [1].

The descending central series of P_u is defined as usual by $\mathcal{C}^0 P_u := P_u$ and $\mathcal{C}^{i+1} P_u := (\mathcal{C}^i P_u, P_u)$ for $i \geq 0$. Since P_u is nilpotent, the smallest integer m such that $\mathcal{C}^m P_u = \{e\}$ is the class of nilpotency of P_u, i.e., the *length* of this series, and is also denoted by $\ell(P_u)$.

Throughout, we use the labeling of the Dynkin diagram of G (i.e. of Σ) as well as the notation for roots in systems of exceptional type as in [3, Planches I - IX]. Our general reference for algebraic groups is Borel's book [2].