ON NORMAL ABELIAN SUBGROUPS IN PARABOLIC GROUPS

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Abstract. Let G be a reductive algebraic group, P a parabolic subgroup of G with unipotent radical P_u, and A a closed connected unipotent subgroup of P_u which is normalized by P. We show that P acts on A with finitely many orbits provided A is abelian. This generalizes a well-known finiteness result, namely the case when A is central in P_u. We also obtain an analogous result for the adjoint action of P on invariant linear subspaces which are abelian as subalgebras of the Lie algebra of P_u. Finally, we discuss a connection to some work of Mal'cev on abelian subalgebras of the Lie algebra of G.

1. Introduction

Throughout, G denotes a (connected) reductive algebraic group defined over an algebraically closed field k of characteristic $p \geq 0$ and P is a parabolic subgroup of G with unipotent radical P_u. The aim of this note is the following result.

Theorem 1.1. *Let G be a reductive algebraic group, P a parabolic subgroup of G, and A a closed connected normal subgroup of P in P_u. If A is abelian, then P has finitely many orbits on A.*

The particular case when A is in the center of P_u is well-known. Then the action factors through a Levi subgroup of P. In characteristic 0 the finiteness follows from a result of Vinberg [41, §2] on gradings of Lie algebras (see also Kac [15]) and in general from work of Richardson [28, §3]. For a detailed account of the orbit structure in this situation, see [24] and [29, §2, §5].

Observe that for abelian P-invariant sub-factors in P_u, the analogous statement of the theorem is false in general. Indeed, this fact is the basis for constructing entire families of parabolic subgroups which admit an infinite number of orbits on the unipotent radical, or its Lie algebra, e.g., see [25], [26], [30], and [31]. Examples in this context also show that a parabolic subgroup may have an infinite number of orbits on a normal subgroup of nilpotency class two, cf. [9], [16].

The *modality* of the action of P on the normal subgroup A is the maximal number of parameters upon which a family of P-orbits on A depends; likewise for the adjoint action of P on the Lie algebra of A, cf. [26]. The basic machinery for investigating the modality of parabolic subgroups of reductive groups was introduced in [26]. Apart from [26] there are several recent articles related to this subject. For instance, all parabolic subgroups P of classical algebraic groups with a finite number of orbits on P_u are determined in [12] and [13]. Similar results for exceptional groups are obtained in [14]. In [8] all such P in $GL_n(k)$ are classified with a finite number of orbits on a given term of the lower central series of P_u. More generally, in [25] and [32] the modality of the action of P on the Lie algebra of P_u is investigated for any reductive G.

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The proof of Theorem 1.1 readily reduces to the case when G is simple, P is a Borel subgroup B of G, and A is a maximal closed connected normal abelian subgoup of B. In Section 3 we classify all such A, up to G-conjugacy (Theorem 3.1), and in Section 5 we show in each instance that B acts on A with a finite number of orbits.

Our proof of Theorem 1.1 uses an extension of the classification of spherical Levi subgroups of reductive groups to arbitrary characteristic due to Brundan [7, §4] (see Section 4). From that we immediately obtain a proof of those cases of Theorem 1.1 where A is contained in the unipotent radical of a parabolic subgroup whose Levi factor is spherical in G. In the cases where we cannot appeal to spherical Levi subgroups directly, a construction from [1] allows us to apply these results partially.

In the two final sections we discuss the situation for the adjoint action of P on abelian ideals in the Lie algebra of P as well as a connection between Theorem 3.1 and Mal'cev's classification of abelian subalgebras of the Lie algebra of G of maximal dimension [21].

Both, our classification of the maximal closed connected normal abelian subgroups A of B, as well as the fact that in each of these instances B operates on A with a finite number of orbits, are obtained in case studies. It would be highly desirable to have a uniform proof of Theorem 1.1 free of case analysis, even for the expense of some characteristic restrictions.

2. Notation and Preliminaries

We denote the Lie algebra of G by $\operatorname{Lie} G$ or $\mathfrak{g}$ and the identity element of G by e; likewise for subgroups of G. Let T be a fixed maximal torus in G and $\Psi = \Psi(G)$ the set of roots of G with respect to T. Fix a Borel subgroup B of G containing T and let $\Sigma = \{\sigma_1, \sigma_2, \dots\}$ be the set of simple roots of Ψ defined by B such that the positive integral span of Σ in Ψ is $\Psi^+ = \Psi(B)$. The highest (long) root in Ψ is denoted by ϱ. If all roots in Ψ are of the same length, they are all called long. A subset I of Ψ^+ is an *ideal* in Ψ^+ ([38, p. 24]) provided I is closed under addition by elements from Ψ^+. For a root β of G we denote by U_β the corresponding one-parameter unipotent subgroup of G normalized by T, and the root subspace $\operatorname{Lie} U_\beta$ of $\mathfrak{g}$ by $\mathfrak{u}_\beta$. The members $U_\beta(\xi)$, where $\xi \in k$, of U_β are called *root elements*.

Suppose that G is simple (over its center). A prime is said to be *bad* for G if it divides the coefficient of a simple root in ϱ, else it is called *good* for G [37, §I,4]. Furthermore, we say that a prime is *very bad* for G if it divides a structure constant of the Chevalley commutator relations for G. Thus, if p is very bad for G, there are degeneracies in these relations. This only occurs if $p = 2$ and G is of type B_r, C_r, F_4, or G_2, or $p = 3$ and G is of type G_2. The same notions apply to reductive groups by means of simple components [39, 3.6].

We may assume that each parabolic subgroup P of G considered contains B.

Let N be a closed connected normal subgroup of P in P_u. Since N is normalized by $T \subset P$, i.e., N is T-regular [11], the root spaces of $\mathfrak{n}$ relative to T are also root spaces of $\mathfrak{g}$ relative to T, and the set of roots of N with respect to T, denoted by $\Psi(N)$, is a subset of Ψ. Suppose that $\Psi(N)$ is closed under addition in Ψ. Note that this is automatically satisfied provided p is not very bad for G. Then $\mathfrak{n} = \oplus \mathfrak{u}_\beta$ ($\beta \in \Psi(N)$) and consequently, $N = \prod U_\beta$, where the product is taken in some fixed order over $\Psi(N)$. The *support* of an element x in N, denoted by $\operatorname{supp} x$, consists of all roots β for which the projection $N \to U_\beta$ is nontrivial when evaluated at x.