A PURITY THEOREM FOR THE WITT GROUP

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Abstract. Let A be a regular local ring and K its field of fractions. We denote by W the Witt group functor that classifies quadratic spaces. We say that purity holds for A if $\mathrm{W}(A)$ is the intersection of all $\mathrm{W}(A_{\mathfrak{p}}) \subset \mathrm{W}(K)$, where $\mathfrak{p}$ runs over the height one prime ideals of A. We prove that purity holds for every regular local ring containing a field of characteristic $\neq 2$.

Résumé. Soit A un anneau local régulier et K son corps des fractions. Soit W le foncteur groupe de Witt qui classifie les espaces quadratiques. On dit que le théorème de pureté vaut pour A si $\mathrm{W}(A)$ est l'intersection de tous les $\mathrm{W}(A_{\mathfrak{p}}) \subset \mathrm{W}(K)$, où $\mathfrak{p}$ parcourt les idéaux premiers de hauteur égale à 1 de A. Nous démontrons que le théorème de pureté vaut pour tout anneau local régulier qui contient un corps de caractéristique $\neq 2$.

1. Introduction

We briefly review the definitions of quadratic spaces and Witt groups. A very detailed exposition of these topics may be found in [**8**] and in [**9**].

Let X be a scheme such that 2 is invertible in $\Gamma(\mathcal{O}_X)$. A *quadratic space* over X is a pair $\mathbf{q} = (\mathcal{E}, q)$ consisting of a locally free coherent sheaf (we also say "vector bundle") $\mathcal{E}$ and a symmetric isomorphism $q : \mathcal{E} \to \mathcal{E}^* = \mathrm{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$: this means that, after identifying $\mathcal{E}$ with $\mathcal{E}^{**}$ in the usual way, it satisfies $q = q^*$.

An *isometry* $\varphi : \mathbf{q} \to \mathbf{q}'$ is an isomorphism $\varphi : \mathcal{E} \to \mathcal{E}'$ such that the square

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\varphi"] \arrow[d, "q"'] & \mathcal{E}' \arrow[d, "q'"] \\ \mathcal{E}^* & \mathcal{E}'^* \arrow[l, "\varphi^*"] \end{tikzcd}$$

commutes.

The *orthogonal sum* of $\mathbf{q}$ and $\mathbf{q}'$ is the space $\mathbf{q} \perp \mathbf{q}' = (\mathcal{E} \oplus \mathcal{E}', q \oplus q')$.

Let $\mathbf{q} = (\mathcal{E}, q)$ be a quadratic space over X and $\mathcal{F}$ a subsheaf of $\mathcal{E}$. The *orthogonal* $\mathcal{F}^\perp$ of $\mathcal{F}$ is the kernel of $q \circ i^*$, where i denotes the inclusion of $\mathcal{F}$ into $\mathcal{E}$.

A subbundle $\mathcal{L}$ of $\mathcal{E}$ is a *sublagrangian of* $\mathbf{q}$ if $\mathcal{L} \subseteq \mathcal{L}^\perp$, and it is a *lagrangian* if $\mathcal{L} = \mathcal{L}^\perp$. Note that lagrangians and sublagrangians are subbundles, i.e. locally direct factors, not just subsheaves. A space $\mathbf{q} = (\mathcal{E}, q)$ is said to be *metabolic* if it has a lagrangian.

Let $GW(X)$ denote the Grothendieck group of quadratic spaces over X with respect to the orthogonal sum. Let M be the subgroup of $GW(X)$ generated by metabolic spaces. The *Witt group of* X is the quotient $\mathrm{W}(X) = GW(X)/M$. If $f : X \to Y$ is a map of schemes and $\mathbf{q} = (\mathcal{E}, q)$ is space over Y, the pair $f^*\mathbf{q} = (f^*\mathcal{E}, f^*q)$ is a quadratic space over X. It is easily seen that f^* respects orthogonal sums and maps metabolic spaces to metabolic spaces, thus f induces a group homomorphism $\mathrm{W}(f) : \mathrm{W}(Y) \to \mathrm{W}(X)$ and W turns out to be a contravariant functor from the category of schemes to the category of abelian groups.

If $X = \mathrm{S}pec(A)$ is affine, a quadratic space over X is the same as a pair (P, q) consisting of a finitely projective A-module P and an A-linear isomorphism $q : P \to P^*$ such that $q = q^*$. In this case a space (P, q) is metabolic if and only if it is isometric to a space of the form $\left(L \oplus L^*, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$.

For an affine scheme $X = \mathrm{S}pec(A)$ we denote $\mathrm{W}(X)$ by $\mathrm{W}(A)$.

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Let now X be an integral scheme and $K = k(X)$ its field of rational functions. By the functoriality of W there is a canonical map $\mathrm{W}(X) \to \mathrm{W}(K)$ and, for every point $x \in X$, a canonical map $\mathrm{W}(\mathcal{O}_{X,x}) \to \mathrm{W}(K)$. We say that an element $\xi \in \mathrm{W}(K)$ is *unramified at* x if ξ is in the image of $\mathrm{W}(\mathcal{O}_{X,x})$. We say that an element $\xi \in \mathrm{W}(K)$ is *unramified (over X)* if it is unramified at every height one point $x \in X$. We say that *purity holds for* X if every unramified element of $\mathrm{W}(K)$ belongs to the image of $\mathrm{W}(X)$ in $\mathrm{W}(K)$.
Purity is known to hold for every regular integral noetherian scheme of dimension at most two [**3**] and for every regular integral noetherian affine scheme of dimension three [**14**].
The main result of this paper is the following purity theorem (§7).

Theorem A. *Purity holds for any regular local ring containing a field of characteristic $\neq 2$.*

Theorem A will be deduced from the same statement for essentially smooth local algebras over a field, using a well-known result of Dorin Popescu.
Further, using essentially the same methods, we prove (§8)

Theorem B. *Let A be a regular local ring containing a field of characteristic $\neq 2$ and K the field of fractions of A. Let f be a regular parameter of A. The natural homomorphism $\mathrm{W}(A_f) \to \mathrm{W}(K)$ is injective.*

¿From this, using a result of Piotr Jaworski for 2-dimensional regular rings, we deduce (§9)

Theorem C. *Let A be a regular local ring containing a field of characteristic $\neq 2$ and f a regular parameter of A. There is a short exact sequence*

$$0 \longrightarrow \mathrm{W}(A) \longrightarrow \mathrm{W}(A_f) \xrightarrow{\delta} \mathrm{W}(A/Af) \longrightarrow 0\,,$$

where δ is the restriction to $\mathrm{W}(A_f)$ of the second residue homomorphism ∂_f at the height one prime $\mathfrak{p} = Af$.

Let $A((t)) = A[[t]]_t$ be the ring of formal Laurent series over A. As a special case of Theorem C we can formulate (§9)

Theorem D. *Let A be a regular local ring containing a field of characteristic $\neq 2$. There exists a split short exact sequence*

$$0 \to \mathrm{W}(A) \to \mathrm{W}(A((t))) \to \mathrm{W}(A) \to 0\,.$$

Remark. The method used for proving purity for an essentially smooth local k-algebra A also yields a new proof of the injectivity of $\mathrm{W}(A)$ into the Witt group $\mathrm{W}(K)$ of its field of fractions. Since this result is well-known and not very difficult (see for instance [**13**]) we use it whenever it is convenient, without proving it again.
Our proof has been inspired by Vladimir Voevodsky's work [**19**] and makes essential use of a non-degenerate trace form for finite extensions of smooth algebras, which was discovered by Leonhard Euler in a special case. We recall its definition and main properties in §§ 2 and 3.

2. The Euler trace

Let k be any field and $A \hookrightarrow B$ a finite extension of smooth, purely d-dimensional k-algebras. Let Ω_A and Ω_B be the modules of Kähler differentials of A and B over k and let $\Omega_{B/A}$ be the module of relative differentials of B over A. Let $\omega_A = \bigwedge^d \Omega_A$, $\omega_B = \bigwedge^d \Omega_B$.

Proposition 2.1. *There exists an isomorphism of B-modules*

$$\omega_B \xrightarrow{\sim} \mathrm{H}om_A(B, \omega_A)$$

.

Proof. Let R be the polynomial algebra $A[X_1, \ldots, X_n]$ and $\rho : R \to B$ a surjective homomorphism of A-algebras. Let $I = \ker(\rho)$. Since B is a local complete intersection over A, by Lemma 4.4 of [**17**] there exists an isomorphism of B-modules

$$(*) \qquad \mathrm{H}om_A(B, A) \simeq \bigwedge^n \left(\mathrm{H}om_B(I/I^2, B)\right)\,.$$