ON MINIMAL APPROXIMATIONS OF MODULES

HENNING KRAUSE AND MANUEL SAORÍN

Let R be a ring and consider the category $\operatorname{Mod} R$ of (right) R-modules. Given a class $\mathcal{C}$ of R-modules, a morphism $M \to N$ in $\operatorname{Mod} R$ is called a left $\mathcal{C}$-approximation of M provided that N belongs to $\mathcal{C}$ and the induced morphism $\operatorname{Hom}_R(N, C) \to \operatorname{Hom}_R(M, C)$ is surjective for every C in $\mathcal{C}$. This concept has been introduced by Auslander and Smalø [2], and independently by Enochs [5]. Approximations arise naturally in various situations, for instance in representation theory of finite dimensional algebras, in particular in tilting theory, or in commutative algebra. In this note we address the problem of finding minimal approximations. In general, a left $\mathcal{C}$-approximation $M \to N$ of M is not uniquely determined by M, but it happens quite often that there exists an approximation $\phi\colon M \to N$ which is left minimal in the sense that any endomorphism ψ of N satisfying $\psi \circ \phi = \phi$ is an isomorphism. Note that such a minimal approximation of M is unique up to isomorphism.

This paper is divided into four sections. In Section 1 we characterize the morphisms $\phi\colon M \to N$ which admit a decomposition $\phi = (\phi', \phi'')\colon M \to N' \coprod N'' = N$ such that ϕ' is left minimal and $\phi'' = 0$. For example, such a decomposition always exists if N is pure-injective. In Section 2 we discuss criteria for a class $\mathcal{C}$ of R-modules such that every R-module admits a left $\mathcal{C}$-approximation. In Section 3 we introduce a new class of modules M which we call product-complete since the class $\operatorname{Add} M$ of direct summands of coproducts of copies of M is closed under taking products. We prove that an R-module M is product-complete if and only if every R-module admits a (minimal) left $\operatorname{Add} M$-approximation. In fact, we provide a host of characterizations of such modules which indicate that product-complete modules are particulary well-behaved. For example, every endofinite module is product-complete. More precisely, we prove in Section 4 that a module M is endofinite if and only if every direct summand of M is product-complete.

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1. Left minimal morphisms

Let R be a an associative ring with identity. We consider the category $\operatorname{Mod} R$ of (right) R-modules and denote by $\operatorname{mod} R$ the full subcategory of finitely presented R-modules. A morphism $\phi\colon M \to N$ in any category is said to be *left minimal* if any endomorphism ψ of N satisfying $\psi \circ \phi = \phi$ is an isomorphism. A concept which

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is closely related is that of an injective envelope. Recall that a monomorphism $\phi\colon M \to N$ in any abelian category is an *injective envelope* of M provided that N is an injective object and every morphism $N \to N'$ is a monomorphism if the composition with ϕ is a monomorphism.

Lemma 1.1. *A monomorphism $\phi\colon M \to N$ into an injective object N is an injective envelope if and only if ϕ is left minimal.*

Proof. Straightforward. □

The following result gives the precise relation between left minimal morphisms and injective envelopes.

Proposition 1.2. *Let $\phi\colon M \to N$ be a morphism in some additive category $\mathcal{A}$ and suppose that idempotents in $\operatorname{End}(N)$ split. Then the following are equivalent:*

(1) *There exists a decomposition $\phi = (\phi', \phi'')\colon M \to N' \coprod N'' = N$ such that ϕ' is left minimal and $\phi'' = 0$.*
(2) *There exists an abelian category $\mathcal{B}$ and an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ such that*
   (a) *$F(N)$ is injective;*
   (b) *F induces an isomorphism $\operatorname{Hom}(M \coprod N, N) \to \operatorname{Hom}(F(M \coprod N), F(N))$;*
   (c) *the image $\operatorname{Im} F(\phi)$ has an injective envelope.*

A decomposition $\phi = (\phi', \phi'')\colon M \to N' \coprod N'' = N$ of a morphism ϕ such that ϕ' is left minimal and $\phi'' = 0$ is unique up to isomorphism. The morphism ϕ' is sometimes called the *minimal version* of ϕ.

Proof. We apply the characterization of injective envelopes from Lemma 1.1.

(1) $\Rightarrow$ (2) Let $S = \operatorname{End}(N)^{\mathrm{op}}$ and consider the functor

$$F\colon \mathcal{A} \longrightarrow \mathcal{B} = (\operatorname{Mod} S)^{\mathrm{op}}, \quad X \mapsto \operatorname{Hom}(X, N).$$

It is clear that (a) and (b) are satisfied. A decomposition $\phi = (\phi', \phi'')\colon M \to N' \coprod N'' = N$ of ϕ such that ϕ' is left minimal and $\phi'' = 0$ induces a morphism $\operatorname{Im} F(\phi) \to F(N')$ in $\mathcal{B}$ which is an injective envelope by Lemma 1.1.

(2) $\Rightarrow$ (1) Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor and suppose that $\varepsilon\colon \operatorname{Im} F(\phi) \to Q$ is an injective envelope. Consider the induced morphism $\psi\colon \operatorname{Im} F(\phi) \to F(N)$. There are morphisms $\alpha\colon Q \to F(N)$ and $\beta\colon F(N) \to Q$ satisfying $\alpha \circ \varepsilon = \psi$ and $\beta \circ \psi = \varepsilon$ since Q and $F(N)$ are injective. We may assume that $\beta \circ \alpha = \mathrm{id}_Q$ since ε is an injective envelope. Thus $\alpha \circ \beta$ is an idempotent in $\operatorname{End}(F(N))$ and we obtain therefore a decomposition $N = N' \coprod N''$ in $\mathcal{A}$ with $F(N') = \operatorname{Im} \alpha$. It is now easily checked that in the induced decomposition $\phi = (\phi', \phi'')\colon M \to N' \coprod N'' = N$ of ϕ the morphism ϕ' is left minimal and $\phi'' = 0$. □

Our main application of the preceding result depends on the well-known fact that the fully faithful functor

$$\operatorname{Mod} R \longrightarrow D(R) = (\operatorname{mod} R^{\mathrm{op}}, \mathrm{Ab}), \quad M \mapsto M \otimes_R -$$

from $\operatorname{Mod} R$ into the category of additive functors $\operatorname{mod} R^{\mathrm{op}} \to \mathrm{Ab}$ identifies the pure-injective R-modules with the injective objects in $D(R)$. Recall that a map