ON COCOMMUTATIVE HOPF ALGEBRAS OF FINITE REPRESENTATION TYPE

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Abstract

Let $\mathcal{G}$ be a finite algebraic group, defined over an algebraically closed field k of characteristic $p > 0$. Such a group decomposes into a semidirect product $\mathcal{G} = \mathcal{G}^0 \rtimes \mathcal{G}_{red}$ with a constant group $\mathcal{G}_{red}$ and a normal infinitesimal subgroup $\mathcal{G}^0$. If the principal block $\mathcal{B}_0(\mathcal{G})$ of the group algebra $H(\mathcal{G})$ has finite representation type, then both constituents have the same property, with at least one of them being semisimple. We determine the structure of the infinitesimal constituent $\mathcal{G}^0$ up to the classification of V-uniserial groups.

0. Introduction

This paper is concerned with the representation theory of finite-dimensional cocommutative Hopf algebras over algebraically closed fields of positive characteristic. As is well-known, such an algebra can be viewed as the group algebra of a finite algebraic k-group $\mathcal{G}$. Special cases are the Hopf algebras associated to constant groups, i.e., the modular group algebras, as well as those of the infinitesimal groups of height ≤ 1, that is, restricted enveloping algebras of restricted Lie algebras. The representation theory of both of these classes has received considerable attention. In either case one has fairly detailed information on the structure of the representation-finite and tame Hopf algebras (cf. [6, 8, 10, 15, 23, 25, 31]). Our ultimate goal will be the extension of these results to arbitrary cocommutative Hopf algebras.

The main results of this paper, Theorems 3.6 and 4.1, provide a reduction to the classification of the so-called *V-uniserial* unipotent groups. The solution to this problem involves different techniques and is therefore relegated to a separate paper.

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Our classification proceeds progressively, starting with infinitesimal groups of height ≤ 1. Here the main tool are the rank varieties of modules, that were introduced by Carlson [4] for finite groups and Friedlander-Parshall [12] for infinitesimal groups of height ≤ 1. However, even in the context of infinitesimal groups, these varieties still provide enough information to narrow down the list of representation-finite cocommutative Hopf algebras.

Following a few technical preparations, we determine in section 2 the representation-finite blocks of reduced enveloping algebras. This class of Frobenius algebras, which plays an important rôle in the representation theory of restricted Lie algebras, encompasses the Hopf algebras of infinitesimal groups of height ≤ 1. From our result, Theorem 2.2, one obtains the classification of restricted Lie algebras of finite representation type, which turns out to be crucial for the general classification. Theorem 3.6 provides the structure of those infinitesimal groups, whose principal blocks are representation-finite. In §4 we address the general case by considering representations of finite algebraic groups. These are semidirect products of a constant and an infinitesimal part, allowing us to view the Hopf algebra as a skew group algebra over its infinitesimal component. It is shown that such a Hopf algebra is representation-finite if and only if both constituents have this property, with at least one of them being semisimple.

In retrospect it appears that the subject can be approached from two different points of view. The announcements [23, 31] delineate geometric methods that have led to precursors of our results. Homological techniques have proven to be more effective for the purposes of this paper, while geometric arguments will be an indispensible tool in our future work.

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1. Preliminaries

Throughout we will be working over an algebraically closed field k of positive characteristic $p > 0$. Given a finite-dimensional Hopf algebra H, we let $\mathcal{P}(H)$ be the restricted Lie algebra of its primitive elements. According to [21, (6.11)] the subalgebra of H that is generated by $\mathcal{P}(H)$ is a sub-Hopf algebra, which is isomorphic to $u(\mathcal{P}(H))$, the restricted enveloping algebra of $\mathcal{P}(H)$.

In the sequel all H-modules are assumed to be finite-dimensional left H-modules. For an H-module M and a subalgebra $\Lambda \subset H$, we will occasionally denote by $M|_\Lambda$ the restriction of M to Λ.

Definition. Let M be an H-module. Then

$$\mathcal{V}_H(M) := \{x \in \mathcal{P}(H) \ ; \ x^p = 0 \ and \ M|_{\langle x \rangle} \ is \ not \ free\} \cup \{0\}.$$

is called the *rank variety* of M.

Note that $\mathcal{V}_H(M) = \mathcal{V}_{u(\mathcal{P}(H))}(M|_{u(\mathcal{P}(H))})$ is the rank variety of the restricted $\mathcal{P}(H)$-module $M|_{u(\mathcal{P}(H))}$.