Circular Biextensions of Tame Concealed Algebras

Peter Dräxler
Fakultät für Mathematik, Universität Bielefeld
POBox 100131, D-33501 Bielefeld, Germany

1 Introduction and main result

Let k be an algebraically closed field. Associated with the extended Dynkin diagrams $\widetilde{\mathbb{A}}_n$ $(n \geq 1)$, $\widetilde{\mathbb{D}}_n$ $(n \geq 4)$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ there are the tame concealed algebras. They are defined as the endomorphism algebras of basic preprojective tilting modules of hereditary algebras $k\vec{\Delta}$ where the underlying graph Δ of the quiver $\vec{\Delta}$ is one of the extended Dynkin diagrams. If Δ is different from $\widetilde{\mathbb{A}}_n$, then all quivers $\vec{\Delta}$ having Δ as underlying graph yield the same tame concealed algebras B. On the other hand, for $\Delta = \widetilde{\mathbb{A}}_n$ one obtains various classes depending on the number p of clockwise and q of counterclockwise arrows in $\vec{\Delta}$. Hence we define the *extended Dynkin type* of a tame concealed algebra B as Δ if B is the endomorphism algebra of a basic preprojective tilting module over $k\vec{\Delta}$ where Δ is one of the diagrams $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$. Moreover, the extended Dynkin type is defined as $\widetilde{\mathbb{A}}_{p,q}$ if B is the endomorphism algebra of a basic preprojective tilting module over $k\vec{\Delta}$ where Δ is $\widetilde{\mathbb{A}}_n$ and in $\vec{\Delta}$ there are p clockwise and q counterclockwise arrows. If we require $p \geq q$, then the extended Dynkin type is uniquely determined by B.

The tame concealed algebras furnish the simplest class of tame algebras which are not of finite representation type. Recall that a finite-dimensional k-algebra is said to be tame if for any non-negative integer d the indecomposable (left) A-modules of dimension d lie up to isomorphism on finitely many 1-parameter families.

Besides a preprojective and a preinjective component the Auslander-Reiten quiver of a tame concealed algebra B consists of a family $(\mathcal{T}_\lambda)_{\lambda \in \mathbb{P}}$ of standard stable tubes $\mathcal{T}_\lambda$ where the index set $\mathbb{P} = \mathbb{P}^1(k)$ is the projective line over k. Each of the tubes $\mathcal{T}_\lambda$ is an abelian length category having only finitely many simple objects. The number $n(\lambda)$ of these simple objects is called the rank of the tube $\mathcal{T}_\lambda$. Almost each of the tubes $\mathcal{T}_\lambda$ is homogeneous, i.e. $n(\lambda) = 1$.

One-point extensions and more general branch enlargements by simple modules in tubes $\mathcal{T}_\lambda$ led to many examples of tame algebras (see e.g. [Ri2]) and even to the complete classification of strongly simply connected algebras which are tame of polynomial growth (see [Sk]). It is well-known that one-point extensions by modules of length 2 in tubes $\mathcal{T}_\lambda$ lead into the area of tame algebras of non-polynomial growth whereas one-point extensions by modules of length ≥ 3 always are wild (see [Ri1]). Multiple one-point extensions by modules of length 2 (even more general derived cases) were investigated in [DrS]. In some proofs in this paper a new class of tame algebras arose. The aim of this paper is to introduce this class of algebras systematically and to analyze the occurring representation types.

Let us introduce the necessary notation. The objects of length l in a tube $\mathcal{T}_\lambda$ form an orbit under the Auslander-Reiten translation τ of B. Let us fix a module R^λ of length 2 in $\mathcal{T}_\lambda$. For all $i = 1, \dots, n(\lambda)$ we put $R_i^\lambda := \tau^i R^\lambda$. Since $R^\lambda_{n(\lambda)} = R^\lambda$, we can assume that the index set for i is $\mathbb{Z}/n(\lambda)$. For each $i \in \mathbb{Z}/n(\lambda)$ there is a non-zero homomorphism $f_i : R^\lambda_{i+1} \to R^\lambda_i$ which is unique up to a multiple in k and satisfies $f_i f_{i+1} = 0$ for all $i \in \mathbb{Z}/n(\lambda)$.

Let us define for $n \in \mathbb{N}$ the finite-dimensional algebra C_n as the algebra kQ_n/I where Q_n is the quiver in Figure 1 whose vertex set is identified with $\mathbb{Z}/n$. The ideal I of the path algebra kQ_n is generated by the paths $X_{i+1}X_i$ for all $i \in \mathbb{Z}/n$.

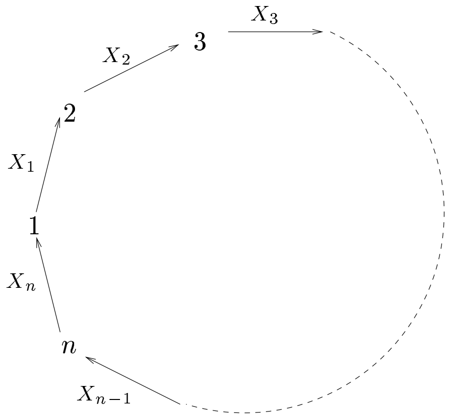

Figure 1: Circular quiver

For $\lambda \in \mathbb{P}$ we obtain a contravariant representation $R(\lambda)$ of $C_{n(\lambda)}$ in $B-$mod by mapping each vertex i of $Q_{n(\lambda)}$ to R_i^λ and each arrow X_i to f_i. But a representation of $C_{n(\lambda)}$ in $B-$mod is nothing but a B-$C_{n(\lambda)}$-bimodule. More general, for any finite subset $T = \{\lambda_1, \dots, \lambda_t\}$ of $\mathbb{P}$ we define the algebra $C(T)$ as the product $C_{n(\lambda_1)} \times \cdots \times C_{n(\lambda_t)}$ and the B-$C(T)$-bimodule $R(T)$ as $\oplus_{i=1}^t R(\lambda_i)$. Given a second finite subset S of $\mathbb{P}$ the *circular biextension* $\boxed{\mathrm{S}}B\boxed{\mathrm{T}}$ of B by the sets S, T is defined to be the matrix algebra

$$\begin{pmatrix} C(T) & 0 & 0 \\ R(T) & B & 0 \\ \mathrm{D}\, R(S)) \otimes_B R(T) & \mathrm{D}\, R(S) & C(S) \end{pmatrix}$$

where the multiplication is given by the usual matrix multiplication and $\mathrm{D} = \mathrm{Hom}_k(-, k)$ is the usual duality. We denote by $\boxed{\mathrm{S}}B$ (resp. $B\boxed{\mathrm{T}}$) the truncated versions where T (resp. S) is the empty set. The algebra $B\boxed{\mathrm{T}}$ will be called *circular extension* of B by T. Dually $\boxed{\mathrm{S}}B$ will be called *circular coextension* of B by S. Now we can formulate the first result of our paper:

Theorem (A). *Let B be a tame concealed algebra and S, T be two finite subsets of $\mathbb{P}$. Then the circular biextension $\boxed{S}B\boxed{T}$ is tame in the following cases:*

(i) The extended Dynkin type of B is $\widetilde{\mathbb{D}}_n$ and $T = \{\lambda\}$, $S = \{\mu\}$ such that $n(\lambda) = n(\mu) = n - 2$.