LATTICE POINT PROBLEMS AND DISTRIBUTION OF VALUES OF QUADRATIC FORMS

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Abstract. For d-dimensional *irrational* ellipsoids E with $d \geq 9$ we show that the number of lattice points in rE is approximated by the volume of rE, as r tends to infinity, up to an error of order $o(r^{d-2})$. The estimate refines an earlier authors' bound of order $\mathcal{O}(r^{d-2})$ which holds for *arbitrary* ellipsoids, and is optimal for rational ellipsoids. As an application we prove a conjecture of Davenport and Lewis that the gaps between successive values, say $s < n(s)$, $s, n(s) \in \mathbb{Q}[\mathbb{Z}^d]$, of a positive definite irrational quadratic form $\mathbb{Q}[x]$, $x \in \mathbb{R}^d$, are shrinking, i.e., that $n(s) - s \to 0$ as $s \to \infty$, for $d \geq 9$. For comparison note that $\sup_s (n(s) - s) < \infty$ and $\inf_s (n(s) - s) > 0$, for rational $\mathbb{Q}[x]$ and $d \geq 5$. As a corollary we derive Oppenheim's conjecture for indefinite irrational quadratic forms, i.e., the set $\mathbb{Q}[\mathbb{Z}^d]$ is dense in $\mathbb{R}$, for $d \geq 9$, which was proved for $d \geq 3$ by G. Margulis in 1986 using other methods. Finally, we provide explicit bounds for errors in terms of certain characteristics of trigonometric sums.

1. Introduction and results

Let $\mathbb{R}^d$, $1 \leq d < \infty$, denote a real d-dimensional Euclidean space with scalar product $\langle \cdot, \cdot \rangle$ and norm

$$|x|^2 = \langle x, x \rangle = x_1^2 + \dots + x_d^2, \qquad \text{for } x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

We shall use as well the norms $|x|_1 = \sum_{j=1}^d |x_j|$ and $|x|_\infty = \max\{ |x_j| : 1 \leq j \leq d\}$. Let $\mathbb{Z}^d$ be the standard lattice of points with integer coordinates in $\mathbb{R}^d$.

For a (measurable) set $B \subset \mathbb{R}^d$, let $\operatorname{vol} B$ denote the Lebesgue measure of B, and let $\operatorname{vol}_{\mathbb{Z}} B$ denote the lattice volume of B, that is the number of points in $B \cap \mathbb{Z}^d$.

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Consider a quadratic form,

$$\mathbb{Q}[x] \stackrel{\text{def}}{=} \langle \mathbb{Q}x, x \rangle, \qquad \text{for} \quad x \in \mathbb{R}^d,$$

where $\mathbb{Q} : \mathbb{R}^d \to \mathbb{R}^d$ denotes a symmetric linear operator with non-zero eigenvalues, say $q_1, \dots, q_d$. Write

$$q_0 = \min_{1 \le j \le d} |q_j|, \qquad q = \max_{1 \le j \le d} |q_j|. \tag{1.1}$$

We assume that the form in non-degenerate, that is, that $q_0 > 0$. Thus, without loss of generality we can and shall assume throughout that $q_0 = 1$, and hence $q \ge 1$.

Define the sets

$$E_s = \big\{ x \in \mathbb{R}^d : \ \mathbb{Q}[x] \le s \big\}, \qquad \text{for} \quad s \in \mathbb{R}.$$

If the operator $\mathbb{Q}$ is positive definite (henceforth called briefly positive), that is, $\mathbb{Q}[x] > 0$, for $x \ne 0$, then E_s is an ellipsoid.

Recall that a quadratic form $\mathbb{Q}[x]$ with a non-zero matrix $\mathbb{Q} = (q_{ij})$, $1 \le i, j \le d$, is rational if there exists an $M \in \mathbb{R}$, $M \ne 0$, such that the matrix $M\mathbb{Q}$ has integer entries only; otherwise it is called irrational. We identify the matrix of $\mathbb{Q}[x]$ with the operator $\mathbb{Q}$.

Our main result—Theorems 2.1 and 2.2 yields the following Theorems 1.1, 1.3 and 1.5 and Corollaries 1.2, 1.4, 1.6 and 1.7, proofs of which we provide in Section 2.

Theorem 1.1. *Assume that $\mathbb{Q}$ is positive and $d \ge 9$. Then*

$$\sup_{a \in \mathbb{R}^d} \Delta(s, \mathbb{Q}, a) \stackrel{\text{def}}{=} \sup_{a \in \mathbb{R}^d} \left| \frac{\operatorname{vol}_{\mathbb{Z}} (E_s + a) - \operatorname{vol} E_s}{\operatorname{vol} E_s} \right| = o(s^{-1}), \tag{1.2}$$

as $s \to \infty$ if and only if $\mathbb{Q}$ is irrational.

The estimate of Theorem 1.1 refines an explicit bound of order $\mathcal{O}(s^{-1})$ obtained by the authors (henceforth called BG) (1994) for *arbitrary* ellipsoids. That result has been proved using probabilistic notions and a version of the basic inequality (see (3.12) below) for trigonometric sums. Some methods of that proof will be used again in this paper. An alternative proof using more extensively the method of large sieves appeared as BG (1997). In the case of *rational* ellipsoids the bound $\mathcal{O}(s^{-1})$ is optimal. For arbitrary ellipsoids Landau (1915) obtained the estimate $\mathcal{O}(s^{-1+1/(1+d)})$, $d \ge 1$. This result has been extended by Hlawka (1950) to convex bodies with smooth boundary and strictly positive Gaussian curvature. Hlawka's estimate has been improved by Krätzel and Nowak (1991, 1992) to $\mathcal{O}(s^{-1+\lambda})$, where $\lambda = 5/(6d+2)$, for $d \ge 8$, and $\lambda = 12/(14d+8)$, for $3 \le d \le 7$. For special ellipsoids a number of particular results is available. For example, the error bound $\mathcal{O}(s^{-1})$ holds for $d \ge 5$ and rational $\mathbb{Q}$ (see Walfisz 1924, $d \ge 9$, and Landau 1924, $d \ge 5$). Jarnik (1928) proved the same bound for diagonal $\mathbb{Q}$ with arbitrary (non zero) real entries. For a discussion see the monograph Walfisz (1957).