On the number of cells of a cellular algebra

by

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1 Introduction

Cellular algebras have been introduced by Graham and Lehrer in [2] in order to discuss the structure of group algebras of symmetric groups and related algebras like certain Hecke algebras (for instance, of type A or B), Brauer algebras, Temperley–Lieb algebras and many others. The definition of a cellular algebra is by the existence of a socalled cell chain of certain ideals (see [3] and section 2 below). Suppose $0 = J_0 \subseteq J_1 \subseteq ... \subseteq J_n = A$ is such a cell chain of ideals of a cellular algebra A. We consider the following two questions:

(1) Is this cell chain a maximal chain, or can it be refined to a longer cell chain?

(2) Is the length n of a maximal cell chain an invariant of the given cellular algebra?

The class of quasi-hereditary algebras (as defined in [1]) has a large intersection with the class of cellular algebras. Thus we recall first that in this case, each heredity chain can be refined to a maximal heredity chain and the length of any maximal heredity chain is an invariant of the algebra, since it is equal to the number of isomorphism classes of simple A–modules. For cellular algebras, a directly analogous statement is the following: The number of J_j occuring in the cell chain with $J_j^2 \not\subseteq J_{j-1}$ is an invariant (see [3]), since again it equals the number of isomorphism classes of simple A–modules.

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The purpose of this note is to answer the above questions and in doing so to provide some further information on cell ideals. More precisely, we show: First, each cell chain is maximal (that is, it cannot be refined any more). However, secondly, the length of a cell chain is not an invariant of a cellular algebra (we provide a counterexample).

2 Cellular algebras

Let us first recall the original definition of cellular algebras in [2] and the equivalent one given in [3].

Definition 2.1 *(Graham and Lehrer [2]) Let R be a commutative Noetherian integral domain. An associative R–algebra A is called a* **cellular algebra** *with cell datum (I, M, C, i) if the following conditions are satisfied:*

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has an R–basis $C^{\lambda}_{S,T}$ where (S, T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is an R–linear anti–automorphism of A with $i^2 = id$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^{\lambda}_{S,T}$ can be written as $(\sum_{U \in M(\lambda)} r_a(U, S) C^{\lambda}_{U,T}) + r'$ where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U, S) \in R$ do not depend on T.

In the following, an R–linear anti–automorphism i of A with $i^2 = id$ will be called an **involution**.

Definition 2.2 *(see [3]) Let A be an R–algebra where R is a commutative Noetherian integral domain. Assume there is an involution i on A. A two–sided ideal J in A is called a* **cell ideal** *if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ is finitely generated and free over R and that there is an isomorphism of A–bimodules $\alpha : J \simeq \Delta \otimes_R i(\Delta)$ (where $i(\Delta) \subset J$ is the i–image of Δ) making the following diagram commutative:*