AN ALTERNATIVE PROOF OF SCHEIDERER'S THEOREM ON THE HASSE PRINCIPLE FOR PRINCIPAL HOMOGENEOUS SPACES

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1. Introduction

Let Y be a smooth irreducible projective curve defined over the real number field $\mathbb{R}$ and $k = \mathbb{R}(Y)$ be the field of $\mathbb{R}$-rational functions on Y. For a point $P \in Y(\mathbb{R})$ we denote the completion of k at the point P by k_P. The present paper is devoted to the Hasse principle for the existence of a rational point on principal homogeneous spaces of a connected linear algebraic group G defined over k. It was Colliot-Thélène who conjectured ([CT], Conjecture 2.9) that for any such space X the Hasse principle holds relative to all local fields k_P, $P \in Y(\mathbb{R})$, i.e. $X(k) \neq \emptyset$ iff $X(k_P) \neq \emptyset$ for each $P \in Y(\mathbb{R})$. Since principal homogeneous spaces of G are in natural one-to-one correspondence with elements of the set $H^1(k, G)$ the latter statement is equivalent to the following: the natural map of pointed sets

$$H^1(k,G) \longrightarrow \prod_{P \in Y(\mathbb{R})} H^1(k_P, G) \tag{1}$$

has trivial kernel ([S]).

In [CT] Colliot-Thélène proved the Hasse principle for algebraic k-tori and reduced the general case to that of a simple simply connected algebraic group G. The case of an arbitrary connected k-group G has been studied by Scheiderer ([Sch1]). To prove the Hasse principle he first made an important observation (which eventually turned out to be crucial); namely that local objects k_P can be replaced by real closures k_ξ of k, $\xi \in \Omega_k$, where Ω_k denotes the set of all orderings of k. Indeed, using the description of orderings of k and the so-called Artin-Lang homomorphism theorem ([Srl], Theorem 3.1) it is easy to show that the condition $X(k_P) \neq \emptyset$ for each real point P on Y implies $X(k_\xi) \neq \emptyset$ for each ordering ξ of k and hence the triviality of the kernel of (1) follows immediately from the triviality of the kernel of

$$\theta : \; H^1(k,G) \longrightarrow \prod_{\xi \in \Omega_k} H^1(k_\xi, G) \tag{2}$$

The question whether θ is injective makes sense not only for the function fields of projective curves but also for an arbitrary field k and it turned out that θ is indeed injective if k has virtual cohomological dimension (vcd) at

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most 1, i.e. $\mathrm{cd}\, k(\sqrt{-1}) \leq 1$ (recall that the function fields in one variable over $\mathbb{R}$ are such). We have even more.

Theorem 1. (Scheiderer, [Sch1]) *Let K be any field of virtual cohomological dimension ≤ 1. Then the Hasse principle holds for any homogeneous K-space X of a connected linear algebraic K-group G.*

Scheiderer's proof can be divided into two parts. In the first one it is proved that for X as in the theorem (here G may even be not connected) there exists a principal homogeneous space Z which is everywhere locally trivial and dominates X. The strategy of the proof in this part going back to Springer ([S], [Sp]) consists of replacing X by a homogeneous space which dominates X and has a smaller stabilizer. It is worth mentioning that in this part most arguments do not use specific properties of K and so most of them are valid over an arbitrary perfect field.

The second part of Scheiderer's proof is devoted to the case of a principal homogeneous space. To treat such a space Scheiderer first constructs a locally constant sheaf of sets $\mathcal{H}^1(G)$ on Ω_K whose stalks are just the sets $H^1(K_\xi, G)$. Then he shows that there exists the natural bijection between the set of global sections of $\mathcal{H}^1(G)$ and $H^1(K,G)$. As a whole the proof in this part is quite long and complicated. It is based on using étale machinery and, in particular, strongly relies on results of the book [Sch2]. The aim of this paper is to provide a simpler and shorter self-contained proof which is based only on the Bruhat-Tits decomposition in semisimple algebraic groups and the so-called strong approximation property (SAP) of fields (see § 3). We show that in fact the Hasse principle follows immediately modulo two facts. Informally speaking one of them says that the kernel of the natural map $H^1(K,T) \to H^1(K,G)$, where G is an (absolutely) simple simply connected linear K-group and T is a K-torus splitting over $K(\sqrt{-1})$, can be parametrised by "good" rational functions (see § 2) and the other says that any field of virtual cohomological dimension ≤ 1 is an SAP field.

2. Algebraic groups splitting over quadratic extensions

Let K be a field and G be an (absolutely) simple simply connected algebraic group of rank n defined over K and splitting over its quadratic extension $L = K(\sqrt{d})$. For simplicity we will also assume in this section that G is K-anisotropic. Consider a Borel L-subgroup B and let

$$\Theta = \mathrm{Gal}\,(L/K) = < \tau \mid \tau^2 = 1 > .$$

Clearly, $T = B \cap \tau(B)$ is a maximal K-defined torus splitting over L. Then the sequence of L-points

$$1 \longrightarrow T(L) \longrightarrow G(L) \longrightarrow (G/T)(L) \longrightarrow 1$$

is exact and hence it induces the exact sequence of pointed cohomology sets

$$(G/T)(K) \xrightarrow{\eta_1} H^1(\Theta, T(L)) \xrightarrow{\eta_2} H^1(\Theta, G(L))$$

Identify the set of cocycles $Z^1(\Theta, T(L))$ with $K^* \times \cdots \times K^*$ and let

$$\eta_3 : K^* \times \cdots \times K^* \longrightarrow H^1(\Theta, T(L)) = K^*/N_{L/K}(L^*) \times \cdots \times K^*/N_{L/K}(L^*)$$

be the canonical map. Denote $s = \dim G - n$.