A characteristic free approach to Brauer algebras

by

Steffen König and Changchang Xi

1 Introduction

Schur–Weyl duality relates the representation theory of the infinite group $GL_n(k)$ with that of the symmetric group Σ_r via the mutually centralising actions of the two groups on the space $(k^n)^{\otimes r}$. Brauer defined the algebras which are now called 'Brauer algebras' (or, slightly incorrect, 'Brauer centraliser algebras') by an analogous situation where GL_n is replaced by either an orthogonal or a symplectic group and the group algebra of the symmetric group is replaced by a Brauer algebra. More precisely, for a fixed integer r and a given base field k, a whole family of Brauer algebras $B_k(r, \delta)$ is defined, depending on a parameter $\delta \in k$, which has to be specialised to certain integers to cover the situation Brauer was interested in.

More recently Brauer algebras and their generalisations, especially the BMW algebras, have been looked at in the context of quantum groups and low–dimensional topology (see [6]).

The classical case of k being the complex numbers (or just having characteristic zero) has been clarified to a large extent by results of Brown, Hanlon

1991 Mathematics Subject Classification: Primary 16D25, 16G30, 20G05. Secondary 57M25, 81R05. *Keywords:* Brauer algebras, orthogonal groups, symplectic groups, cellular algebras.

Both authors have obtained support from the Volkswagen Foundation (Research in Pairs Programme of the Mathematical Research Institute Oberwolfach). S. König also obtained support from Beijing Normal University during his stay in Beijing in May 1997, when most of this paper has been written. C. C. Xi also obtained support from the Young Teacher Foundation of Chinese Educational Committee.

and Wales, and Wenzl: If δ is not an integer of small absolute value, then $B_k(r, \delta)$ is semisimple and its simple representations are known. In the other case of δ being 'singular', Wenzl also obtained a good description of the simple modules. Another such description in the semisimple case has been announced by Kerov. The ring structure of $B_k(r, \delta)$, however, seems to be still unknown even in the characteristic zero case.

Recently, Graham and Lehrer have shown that over any field k, and for any δ, the Brauer algebra $B_k(r, \delta)$ is a cellular algebra. In this way they got a parametrization (which in fact does not depend on k) of the isomorphism classes of simple modules in general. Considering cellular structures thus opens a way for a characteristic free structure theory of Brauer algebras.

The aim of this paper is to work out such a characteristic free approach and to avoid any complicated computation in doing so by using structural properties instead. We start from the scratch and do not use any of the results mentioned above. We first prove that Brauer algebras are cellular (our proof is different from and less computational than that of Graham and Lehrer). More precisely we exhibit in this algebra a structure which we call 'iterated inflation'. As a consequence we do not only get a new proof of the result of Graham and Lehrer, but also of some results of Hanlon and Wales.

Moreover we consider block decompositions of Brauer algebras. Assuming that δ is not singular (but k is arbitrary) we prove in theorem 6.3 that the Brauer algebra is Morita equivalent to a direct product of group algebras of symmetric groups (which correspond to the layers in the iterated inflation). Hence this case is equivalent to the type A case. As a corollary (6.5) we classify under this assumption the blocks of finite representation type of Brauer algebras – these turn out to be Brauer tree algebras.

We organize the paper as follows: In section two we recall the definition of Brauer algebras and the axiomatics of cellular algebras. In sections three and four we consider inflations, which in general are algebras without unit. They occur as pieces of the iterated inflation as which we write the Brauer algebra in section five. More precisely theorem 5.6 tells us that any Brauer algebra can be written as iterated inflation of certain group algebras of symmetric groups (which are explicitly given). This implies the results of Graham and Lehrer. In section six we consider Brauer algebras over a field with nonsingular parameter. In this case, we determine the blocks of the Brauer algebra, and among them those of finite representation type.