A NOTE ON THE QUILLEN-LICHTENBAUM CONJECTURE AND THE ARITHMETIC OF SQUARE RINGS

G.Banaszak, W.Gajda, P.Krasoń and P.Zelewski

Abstract. In this paper we consider the Quillen-Lichtenbaum conjecture for number fields using description of the higher K-theory in terms of SK_1 groups.

1. Introduction

Let F be a number field and let n be a non-negative integer. Fix an odd prime l. We define the wild kernel $WK_n(F)$ as the kernel of the localization map in K-theory of F:

$$\lambda_n : K_n(F) \to \prod_v K_n(F_v).$$

The group $WK_n(F)$ is interesting because we have:

Proposition A. *[Prop. 5.1.] For each $n \geq 0$*

$$WK_n(F) \subseteq K_n(\mathcal{O})_{tor},$$

where $\mathcal{O}$ denotes the ring of integers of F. In particular, the group $WK_n(F)$ is finite.

By [B, theorem 2] there is the following long exact sequence.

$$0 \to K_{2n}^w(\mathcal{O})_l \to K_{2n}(F)_l \to \bigoplus_v W^n(F_v) \to W^n(F) \to 0, \tag{1.1}$$

where v runs over finite places of F. In [B] the group $K_{2n}^w(\mathcal{O})_l$ was called the l-part of the wild kernel for $K_{2n}(F)$. We expect that the groups $WK_{2n}(F)_l$ and $K_{2n}^w(\mathcal{O})_l$ coincide. It is unknown in the moment, since we do not know the K-theory with coefficients in $\mathbb{Z}/l^k$ of a local field in the case when l is divisible by the residue characteristic, except for the case of $\mathbb{Q}_l$ which follows by results of Bökstedt and Madsen [BM].

1991 *Mathematics Subject Classification.* Primary 19F27; Secondary 19Fxx.
This work has been sponsored in part by a KBN grant 2P03A 005 11.

We propose the following conjecture.

Conjecture B. *For all number fields F and for all $n \geq 0$ we have:*

$$WK_n(F)_l = [divK_n(F)]_l$$

where $divK_n(F)$ is the group of divisible elements in $K_n(F)$.

In section 5 of the paper we give the following condition for equivalence of the above conjecture with the Quillen-Lichtenbaum conjecture for the number field F [DF, p. 273].

Theorem C. *[Th. 5.2.]*

(a) *Assume that the following maps:*

$$K_{2m}(F_v)_l \to K_{2m}^{et}(F_v)_l$$

are isomorphisms for all v which divide l and for all $m > 0$. Then the Quillen-Lichtenbaum conjecture for F is equivalent to the conjecture B for all indices $n > 1$.

(b) *If $F = \mathbb{Q}$, then the Quillen-Lichtenbaum conjecture and conjecture B are equivalent for $n > 1$.*

Theorem D. *[Th. 5.5.] The conjecture B holds true for all number fields F and for $0 \leq n \leq 3$. In addition, it holds true for $n = 4$ and $F = \mathbb{Q}$.*

For another reformulation of the Quillen-Lichtenbaum conjecture see [BZ1, p. 390].

By [BZ2, theorem, p.161] and a result of Dayton and Weibel [DW, Corollary 5.3] we get the following exact sequence involving the wild kernel (see Section 6).

$$0 \to WK_n(F) \to SK_1(F_n) \to \prod_v SK_1(F_{vn}^h).$$

In this way we reformulate the conjecture B in terms of elementary linear algebra, number fields and algebraic properties of the square rings F_n. In the last section of the paper we investigate $WK_n(F)$ using the description of the higher K-theory in terms of SK_1.

Notation

(1) We fix an odd rational prime l.
(2) Let F be a number field and let $\mathcal{O}$ be its ring of integers.
(3) Let v be a finite place of $\mathcal{O}$ and let k_v be the residue field for v.
(4) Put F_v to be the completion of F at v and $\mathcal{O}_v$ to be the ring of integers of F_v.
(5) Put $\mathcal{O}_v^h$ to be the henselization of $\mathcal{O}_F$ at v and F_v^h to be the quotient field of $\mathcal{O}_v^h$.
(6) For a commutative ring with identity R and for $n > 0$ we denote by R_n the square ring:

$$R[t_1, t_2, ..., t_n]/((t_1^2 - t_1)...(t_n^2 - t_n)).$$