A characterization of the finite soluble groups.

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1 Introduction.

Let G be a group and M a subgroup of G. Then we say that a subgroup A of G is a *supplement* to M in G, if $G = MA$. In 1965 O. H. Kegel [Ke] introduced the condition

(MC) Every maximal subgroup of G admits a supplement which is cyclic and of prime-power order.

and proved that every finite group G which satisfies condition (MC), is soluble [Ke, Proposition 1].

In the same paper Kegel conjectured that any finite group G whose maximal subgroups admit an abelian supplement is soluble.

Proposition 1 of [Ke] was generalized by R. M. Guralnik in a different way. He showed that if G is a finite group such that every maximal subgroup has prime power index, then $G/O_\infty(G) \cong 1$ or $L_2(7)$ ($O_\infty(G)$ is the maximal normal soluble subgroup of G) [Gu, Corollary 3].

In their paper [BR] J. C. Beidleman and D. J. S. Robinson considered finite groups G having the property that for every proper subgroup H of G there is a cyclic group $X = X(H)$ such that $XH = HX$. They proved that every group G enjoying this property is soluble [BR, Theorem A]. See also the related work by Z. Yaoqing [Ya] and by W. E. Deskins [De].

The proofs of Corollary 3 of [Gu] and of Theorem A of [BR] are based on the classification of finite simple groups.

Another characterization of soluble groups was obained by Y. Wang. He showed that a finite group G is soluble if and only if for every maximal subgroup H of G there exists a normal subgroup $N = N(H)$ of G such that $G = HN$ and $H \cap N \leq Core(H)$ [Wa, Theorem 3.1].

In this paper we will prove the conjecture of Kegel. Notice, that the proof of the conjecture implies Theorem A of [BR] as well as Theorem 3.1 of [Wa].

Let (MA) be the condition

(MA) Every maximal subgroup of G admits an abelian supplement.

We show

Theorem 1.1 *A finite group G which satisfies condition (MA) is soluble.*

Hence Theorem 1.1 proves the conjecture of Kegel. The theorem may be formulated in terms of permutation groups as follows.

Theorem 1.2 *Suppose that G is a finite group such that for any primitive action (G, Ω) of G, there exists an abelian subgroup of G which acts transitively on Ω. Then G is soluble.*

Notice, that a group G satisfying the condition that for any primitive permutation action (G, Ω) of G, there exists an abelian subgroup of G which acts transitively on Ω, is not necessarily soluble. For example let G be an extension of an elementary abelian group N by a non soluble group. Then G is not soluble, but N is a supplement to every maximal subgroup of G which does not contain N.

Notice also, that condition (MA) does not characterize the finite soluble groups as the following example shows.

Example. Let $G = SL_2(3) \cong Q_8 : \mathbf{Z}_3$. Let $Z = Z(O_2(G))$ and let T be a subgroup of order 3 of G. Then $M = ZT$ is maximal in G, but does not have an abelian supplement.

In order to characterize the soluble groups among the finite groups we introduce the condition.

(MC) Every maximal subgroup M of G admits a supplement whose commutator subgroup is contained in M.

Theorem 1.3 *A finite group G is soluble if and only if (MC) holds.*

Moreover we will determine the structure of the groups satisfying property (MN).

(MN) Every maximal subgroup M of G admits a nilpotent supplement.

We will show

Theorem 1.4 *Let G be a finite group which satisfies (MN). Then every composition factor of G is either soluble or isomorphic to $L_2(7)$. Moreover, $G/O_\infty(G)$ is isomorphic to 1 or $L_2(7)$.*

We also determine the groups with property

(MF) Every maximal subgroup of G admits a soluble supplement.

We proof:

Theorem 1.5 *The chief factors of a finite group G which satisfy (MF) are either p-groups or isomorphic to $L_2(q)$, $q \in \{5, 7, 9, 11\}$.*

But these groups still deserve a further analysis.

In the next section we prove that Theorem 1.5 holds for almost simple groups and that there is no almost simple group satisfying (MA) or (MC). These results are obtained by checking the list of maximal factorizations of the almost simple groups by M. Liebeck, C. Praeger and J. Saxl [LPS]. They determine for every almost simple group G the maximal subgroups A and B such that $soc(G)$ is neither a subgroup of A nor of B and such that $G = AB$. This paper of Liebeck, Praeger and Saxl [LPS] is based on the classification of the finite simple groups. Of course, it would be good, if one could prove the Lemmas 2.1 and 2.2 without quoting the classification. In the third section we prove Theorems 1.1, 1.3 and 1.5. The idea of the proofs of these theorems is to show first that the assertions hold if G is a wreath product $W = L \wr S_n$ and then to reduce the general case, that is G a group satisfying (MA) (resp. (MC) or (MF)) to the special situation that G is a wreath product. In the last section we prove Theorem 1.4.

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