THE HILBERT-POINCARE SERIES FOR SOME ALGEBRAS OF INVARIANT OF CYCLIC GROUPS.

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1. Introduction

Let G be a finite group and let $\rho : G \to GL(V)$ be a linear representation into a finite dimensional vector space V over a field K (here K is an algebraically closed field of characteristic zero). Then the group $\rho(G)$ acts naturally on the symmetric algebra $S = S(V)$. Let $R_\rho = S^{\rho(G)}$ be the corresponding algebra of invariants (i.e. $R_\rho = \{s \in S \mid \rho(\sigma)s = s$ for every $\sigma \in G\}$. Then R_ρ is a graded algebra $= \sum_{n=0}^{\infty} R_{\rho,n}$ (here $R_{\rho,n}$ is the homogeneous component of the degree n). The Hilbert–Poincare series of R_ρ (see, [St1,2])

$$P_\rho(t) = \sum_{n=0}^{\infty} (\dim R_{\rho,n}) t^n$$

defines a rational function $\Psi(t)/\Theta(t)$ (we will identify $P_\rho(t)$ and the function $\Psi(t)/\Theta(t)$). We can take different presentations of the function $P_\rho(t)$ in the form $\Psi(t)/\Theta(t)$. In particular, we can take Ψ and Θ such that $\Psi(t) \in Z[t]$, $\Theta(t) = \prod_j (1 - t^{m_j})$ (see, [St 1,2]). In the cases when R_ρ is a **complete intersection** the numerator $\Psi(t)$ of $P_\rho(t)$ can also be written the form $\prod_i (1 - t^{e_i})$. Note, this statement is true for every graded complete intersection but not necessary for an algebra of invariants. R.Stanley formulated a question in [St1]: under what additional conditions the inverse statement holds, i.e. if the numerator of a Cohen-Macaulay graded algebra is of the form $\prod_i (1 - t^{e_i}) / \prod_j (1 - t^{m_j})$ for some $\{e_i\}$, $\{m_j\} \subset N$ then this algebra is a complete intersection. He pointed out two special cases when it is true and gave an example when it is false. It seems it would be reasonable to look at a naturally defined class of graded algebras, say, for algebras of invariants from the point of view of the Stanley's question. However, it is a difficult problem even for the case when G is a cyclic group. Here we give an affirmative answer for the Stanley's question in the cases when G is a cyclic group and ρ is "near" to the regular representation.

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Let $\rho:\ G=\langle\sigma\rangle\to GL(V)$ be a representation of a cyclic group of the order n. Further, let $d\,|\,n$ and let ε_d be a fixed primitive d–th root of unity. For $g\in G$ we put

$$m_d(g) = \textit{multiplicity of the eigenvalue } \varepsilon_d \textit{ of the operator } \rho(g).$$

By m_d we denote $\max\{m_d(g)\,|\,g\in G\}$.

Definition. *We say that a representation ρ is* **almost regular** *if:*
1. $m_n(\sigma)\geq 1$;
2. ρ is $Aut\,G$-invariant (i.e. $\rho(f(\sigma))=\rho(\sigma)$ for every $f\in Aut\,G$);
3. for every divisor d of n the equality $m_d(g)=m_d$ holds if and only if the order of g is equal to d.

Example 1. Let ρ be the regular representation. Then ρ is almost regular. Indeed, conditions 1 and 2 obviously hold. Further, let $g=\sigma^\ell$ where $\ell\,|\,n$. Then the restriction of ρ on the group $\langle g\rangle$ is the sum of ℓ copies of regular representation of $\langle g\rangle$. Hence $m_d(g)=0$ if d is not a divisor of n/ℓ and $m_d(g)=\ell$ if $d\,|\,(n/\ell)$. Thus the maximum of the function $m_d(g)$ is equal to n/d and it is reached only on elements of the order d.

Example 2. Let ρ be a representation satisfying conditions 1 and 2. Further,

$$V=\sum_{d|n} V_d$$

where V_d is a vector space generated by eigenvectors of $\rho(\sigma)$ with eigenvalues which are primitive d–th roots of unity. Suppose the following condition holds:

$3'$. $\dim V_{d_1}<\dim V_{d_2}$ for every divisors $d_1\neq d_2$ of n such that $1\neq d_1\,|\,d_2$.

We show that ρ is almost regular. Let $n=dk\ell$ for some $d,k,\ell,\ d\neq 1,\ k\neq 1$;and let $g=\sigma^\ell$.

Put

$$D_{d,\ell}=\{\,a\,|\,n\ \mid\ \varepsilon_a^\ell \text{ is a primitive } d-\text{th root of unity}\}$$

(here ε_a is a primitive a–th root of unity). Since ρ is Aut G-invariant we get

$$m_d(g)=\sum_{a\in D_{d,\ell}}(\dim V_a)/\varphi(d).$$

If $a\in D_{d,\ell}$ then $\varepsilon_{ak}^{\ell k}$ is a primitive d–th root unity. Using $3'$ we obtain

$$m_d(g^k)\ \geq\ \sum_{a\in D_{d,\ell}}(\dim V_{ak})/\varphi(d)\ >\ m_d(g).$$

Thus the function $m_d(g)$ reaches its maximum only on elements of order d.

Example 3. Let ρ be a representation satisfying condition 1 and 2 and let

$$V=V_n+V_1,$$