RADEMACHER'S THEOREM ON CONFIGURATION SPACES AND APPLICATIONS

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Abstract: We consider an L^2-Wasserstein type distance ρ on the configuration space Γ_X over a Riemannian manifold X, and we prove that ρ-Lipschitz functions are contained in a Dirichlet space associated with a measure on Γ_X satisfying some general assumptions. These assumptions are in particular fulfilled by a large class of tempered grandcanonical Gibbs measures with respect to a superstable lower regular pair potential. As an application we prove a criterion in terms of ρ for a set to be exceptional. This result immediately implies, for instance, a quasi-sure version of the spatial ergodic theorem. We also show that ρ is optimal in the sense that it is the intrinsic metric of our Dirichlet form.

0. Introduction.

Let Γ_X be the configuration space over a Riemannian manifold X. In this paper, we consider a class of probability measures on Γ_X, which in particular contains certain Ruelle type Gibbs measures and mixed Poisson measures. Using a natural 'non-flat' geometric structure of Γ_X, recently analyzed in Albeverio, Kondratiev and Röckner (1996a), (1996b), (1997a), and (1997b), one can define weak derivatives and introduce the related Sobolev spaces. Here we are interested in a more detailed description of this concept of differentiability. Similar to the case of H-differentiability on Wiener space, it turns out that not all values which a function u takes in a small neighborhood of some $\gamma \in \Gamma_X$ are relevant for its weak gradient $\nabla^\Gamma u(\gamma)$, but only those which are located in certain 'directions'. Here, of course, the word 'direction' needs to be defined because of the absence of any linear structure on Γ_X even if $X = \mathbb{R}^d$.

Our way of making the above precise is to prove an infinite dimensional version of the celebrated theorem of Rademacher (1919) stating that Lipschitz functions on $\mathbb{R}^d$ are differentiable almost everywhere and and in the weak sense. On abstract Wiener space and its generalizations, similar results were obtained by Kusuoka (1982a), (1982b), Enchev and Stroock (1993), and Bogachev and Mayer-Wolf (1996). On configuration space, the correct Lipschitz condition is defined through an L^2-Wasserstein type distance function ρ, which, for non-compact X, divides Γ_X into uncountably many disjoint 'fibers', each of the form $\{\omega \mid \rho(\gamma, \omega) < \infty\}$. A consequence of our Rademacher type theorem then is that only the behavior of u in small ρ-balls around γ matters for the value of $\nabla^\Gamma u(\gamma)$.

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In a second result, we prove a partial converse to our Rademacher theorem. It also allows us to identify ρ as the intrinsic metric of certain Dirichlet forms associated with our weak gradient. The resulting variational formula can also be regarded as a Kantorovich-Rubinstein type theorem for our L^2-Wasserstein metric. On abstract Wiener space such a converse to Rademacher's theorem was obtained in Enchev and Stroock (1993) by a different method. As a by-product to our proof we obtain that all measures satisfying our assumptions have full topological support on Γ_X, a result which might be of independent interest, in particular for the Ruelle measures mentioned above.

Another main part of this paper is devoted to applications of the above results to the potential theory on configuration space. In particular, we show that if $A \subset \Gamma_X$ has full measure, then the set of all points with positive ρ-distance to A is exceptional. This, for instance, implies immediately a quasi-sure version of the spatial ergodic theorem on $\Gamma_{\mathbb{R}^d}$. We also give a short proof of the quasi-regularity of our Dirichlet forms.

The organization of the paper is as follows. In Section 1, we describe our set-up and the Rademacher type results under some general conditions on a measure μ on Γ_X. In Section 2, we identify a class of Ruelle type Gibbs measures which satisfy these assumptions. The applications to potential and ergodic theory are presented in Section 3. Sections 4 and 5 are devoted to a detailed analysis of the topological and geometric properties of our L^2-Wasserstein distance ρ. These results serve as a preparation for the proofs of our main theorems, but some may also be of independent interest. Proofs concerning our various Sobolev spaces and Dirichlet forms are given in Section 6. Finally the proofs of our first two theorems are presented in Sections 7 and 8.

1. The Rademacher-type results.

Let Γ_X denote the space of all integer-valued Radon measures on a Riemannian manifold X with Riemannian inner product g. We will assume throughout this paper that (X, g) is smooth, connected and complete. In the case where X is one-dimensional we will assume that X equals $\mathbb{R}$ with its Euclidean metric. Throughout this paper, we will assume that Γ_X is endowed with the topology of vague convergence. Then Γ_X is Polish as a vaguely closed subset of the space of all non-negative Radon measures on X. The set of all $\gamma \in \Gamma_X$ such that $\gamma(\{x\}) \in \{0, 1\}$ is usually called *configuration space* over X, but we will also call Γ_X itself configuration space.

The space Γ_X carries the following geometric structure which was defined in Albeverio, Kondratiev and Röckner (1996), (1997a). The "tangent space" $T_\gamma\Gamma_X$ of Γ_X in some γ is given as $L^2(X \to TX, \gamma)$, i.e., the space of all sections V in the tangent bundle TX of X which are square-integrable with respect to γ:

$$\|V\|^2_{T_\gamma\Gamma_X} := \langle V, V\rangle_{T_\gamma\Gamma_X} := \int g_x(V, V)\, \gamma(dx) < \infty.$$

For ease of notation, we will also use $\|\cdot\|_\gamma$ and $\langle\cdot,\cdot\rangle_\gamma$ instead of $\|\cdot\|_{T_\gamma\Gamma_X}$ and $\langle\cdot,\cdot\rangle_{T_\gamma\Gamma_X}$. By endowing the tangent space $T_\gamma\Gamma_X$ with the inner product $\langle\cdot,\cdot\rangle_\gamma$, Γ_X obtains a Riemannian-type structure, which is is non-trivial (i.e., varies with γ) even when the underlying space X is flat.