Two side bounds on the inverses of diagonally dominant tridiagonal matrices

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Abstract

We establish upper and lower bounds for the entries of the inverses of diagonally dominant tridiagonal matrices. These bounds improve the bounds recently given by Shivakumar and Ji. Moreover, we show how to improve our bounds iteratively. For an $n \times n$ M–matrix this iterative refinement yields the exact inverse after $n-1$ steps.

1 Introduction

Tridiagonal matrices arise in many applications, e.g. the numerical solution of PDE's and cubic spline interpolation. Therefore this class of matrices is studied very well and a lot of theory is known about tridiagonal matrices and their inverses. One of the most beautiful result is due to Gantmacher and Krein [GK1], [GK2] who proved that the inverse of an irreducible symmetric tridiagonal matrix is given by two vectors. A similar result is also proved by Barrett in [Ba]. This result is generalized in several directions. For a review on symmetric tridiagonal matrices and their inverses see Meurant [M].

For many problems it is very helpful to have upper and lower bounds for the entries (or the absolute values of the entries) of the inverse of a matrix. For arbitrary diagonally dominant matrices Ostrowski [O2] already established such bounds. Later on Golub and Meurant gave in [GM] bounds for symmetric positive definite matrices established by using quadrature rules. Similar bounds were established by Robinson and Wathen in [RW]. Decay rates for the entries of inverses of certain banded matrices are established by Demko [D] and several other authors (see Meurant [M] and Nabben [N] and references therein). Of course these rates can also be used as approximations of entries of the inverse of a matrix.

For certain tridiagonal matrices arising in spline interpolation Kershaw derived in [K] upper and lower bounds. Recently, Shivakumar and Ji gave in [SJ] upper and lower bounds for diagonally dominant tridiagonal matrices.

Here, we establish new upper and lower bounds for the entries of the inverses of diagonally dominant tridiagonal matrices which improve the bounds given in [SJ], [K] and [O2]. Moreover, we show how these bounds can be improved iteratively. For an a $n \times n$ M–matrix we obtain after $n-1$ iterations sharp upper bounds for all entries of the inverse, i.e. by improving our bounds we have computed the inverse of the matrix. However, our examples show that one needs just a few iterations to get good bounds. For arbitrary diagonally dominant tridiagonal matrices A we obtain after $n-1$ steps of our iterative refinement the exact inverse of the comparison matrix of A. Moreover, we establish some further results of this iterative refinement for tridiagonal Toeplitz matrices.

This paper is organized as follows. In Section 2 we mention several known result for tridiagonal matrices which we use in the following section. Section 3 contains our main results while Section 4 gives a comparison of our bounds with known results as well as some numerical examples.

2 Preliminaries

We consider real tridiagonal matrices of the form

$$A = \begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{n-2} & a_{n-1} & b_{n-1} \\ & & & c_{n-1} & a_n \end{bmatrix}, \quad c_i, b_i \neq 0. \tag{2.1}$$

Since A might be nonsymmetric we distinguish between diagonally dominance by rows, i.e.

$$|a_i| \geq |b_i| + |c_{i-1}| \quad \text{for all } i = 1, \ldots, n, \tag{2.2}$$