High dimensional knots corresponding to the Fractional Fibonacci groups *

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Abstract

We prove that the natural HNN-extensions of the Fractional Fibonacci groups are fundamental groups of high-dimensional knot complements. We give also some characterization and interpretation of these knots. In particular we shall show that some of them are 2-knots.

1 Introduction

The *fractional Fibonacci groups* $F^{k/l}(2,m)$ were introduced in [6] for integers k, l, m such that $m \geq 3$ and $(k,l) = 1$:

$$F^{k/l}(2,m) \;=\; \langle a_1, a_2, \ldots, a_m \mid a_i^l \, a_{i+1}^k \;=\; a_{i+2}^l, \quad i = 1, \ldots, m \rangle \tag{1}$$

with all indices reduced module m. In the case $k = l = 1$ we get the well-known Fibonacci groups $F(2,m)$ introduced by J. Conway in 1965. The topological and geometric properties of three-dimensional closed orientable manifolds uniformized by groups $F(2,2n)$ and by groups $F^{k/l}(2,2n)$ were

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studied in [2] and [6], respectively. Almost all of these groups (and manifolds) are hyperbolic. Concerning the case m odd, Helling has pointed out to us that the groups $F(2,m)$ have torsion, and it was shown in [7] that these groups cannot be fundamental groups of hyperbolic 3-manifolds or hyperbolic 3-orbifolds.

The investigation of HNN-extensions of the Fibonacci groups as high-dimensional knot groups was started by J. Hillman in [3], where he considered the natural extension of the group $F(2,6)$ that is the fundamental group of the Euclidean Hantzsche-Wendt manifold. This studying was continued in [11], where it was proven that the same HNN extension of the Fibonacci groups $F(2,m)$ also arise as high-dimensional knot groups. In the present paper we shall demonstrate that natural HNN-extensions of the Fractional Fibonacci groups $F^{k/l}(2,m)$ are high-dimensional knot groups if and only if $k=1$.

We recall that a group G is said to be an *n-dimensional knot group*, for $n \geq 1$, if $G = \pi_1(S^{n+2} \setminus K(S^n))$ for some n-dimensional knot $K: S^n \to S^{n+2}$. Kervaire [10, section 11D] obtained the following algebraic characterization of such groups: *A group G is a 3-knot group (and so, n-knot group for any $n \geq 3$) if and only if it is finitely presentable, $H_1(G,\mathbb{Z}) = \mathbb{Z}$, $H_2(G,\mathbb{Z}) = 0$ and G has weight 1 (that is G is the normal closure of some single element).* In dimensions $n = 1,2$ the above conditions are necessary but no longer sufficient. Our studying of the group extensions of $F^{k/l}(2,m)$ will be based on this criterion and on Zeeman's twist spun construction of knots [12].

2 The natural HNN-extension

The group $F^{k/l}(2,m)$ with the presentation (1) is an example of a cyclically presented group in the sense of [5]. We recall that a group G is *cyclically presented* if for some n and w it has a presentation

$$G = \langle x_1, x_2, \ldots, x_m \mid w, \eta(w), \ldots \eta^{m-1}(w) \rangle, \tag{2}$$

where $\eta : \mathbb{F}_m \to \mathbb{F}_m$ is the automorphism of the free group $\mathbb{F}_m = \langle x_1, \ldots, x_m \rangle$ of rank m given by $\eta(x_i) = x_{i+1}, i = 1, \ldots, m$, and $w \in \mathbb{F}_m$ is a cyclically reduced word. The fractional Fibonacci groups with the presentation (1) arise in this construction for $w = x_1^l x_2^k x_3^{-l}$.