ON SUPPORT VARIETIES OF AUSLANDER-REITEN COMPONENTS

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Abstract

Let $u(L,\chi)$ be the reduced enveloping algebra associated to a finite dimensional restricted Lie algebra $(L,[p])$ and a linear form $\chi \in L^*$. It is shown that a connected component Θ of the stable Auslander-Reiten quiver of $u(L,\chi)$ is of type $\mathbb{Z}[A_\infty]$, whenever its support variety $\mathcal{V}_L(\Theta)$ has dimension ≥ 3. Various applications concerning AR-components of Lie algebras of algebraic groups and the structure of hearts of principal indecomposable $u(L,\chi)$-modules are given.

1. Introduction and Preliminaries

In recent work [10] K. Erdmann has shown that the nonperiodic components of the stable Auslander-Reiten quivers belonging to wild blocks of a modular group algebra are isomorphic to $\mathbb{Z}[A_\infty]$. Accordingly, the Auslander-Reiten theory of group algebras is now very well understood. By contrast, relatively little is known about the AR-quivers of the family $(u(L,\chi))_{\chi\in L^*}$ of reduced enveloping algebras associated to a restricted Lie algebra $(L,[p])$. One main problem in this context is the lack of a comprehensive block theory of enveloping algebras. Thus, aside from the possibility of considering special classes of Lie algebras (cf. [11, 16]), Carlson's geometric approach [8], that was transferred to the theory of restricted Lie algebras by Friedlander and Parshall [18], appears most promising. In this vein, one result was given by Rickard [31], who proved that $u(L,\chi)$ is wild whenever it admits an indecomposable module whose support variety has dimension ≥ 3. The principal result of this note, Theorem 2.1, shows that the stable Auslander-Reiten component of such a module has tree class A_∞. In the context of simple Lie algebras our result can be used to identify $s\ell(2)$ as the only algebra whose restricted envelope admits an Auslander-Reiten component of Euclidean tree class. The final section illustrates another application of (2.1) by showing

*Supported by N.S.A. Grant MDA904-96-1-0040 and the D.F.G.

[0]1991 Mathematics Subject Classification: 16G70, 17B50

that the hearts of the principal indecomposable modules of simple Lie algebras $\neq s\ell(2)$ are indecomposable.

Let Λ be a self-injective algebra over a field F. Throughout this paper all modules are assumed to be finite dimensional. The stable Auslander-Reiten quiver $\Gamma_s(\Lambda)$ has the isomorphism types $[M]$ of nonprojective indecomposable Λ-modules as its vertices. There is an arrow $[M] \to [N]$, whenever there exists an irreducible map $M \longrightarrow N$. We denote the Auslander-Reiten translation of $\Gamma_s(\Lambda)$ by τ and refer the reader to [2] concerning basic properties of AR-quivers.

A map $g : M \longrightarrow N$ between two indecomposable Λ-modules is called *properly irreducible* if it is irreducible, but not almost split. The following subsidiary results, due to K. Erdmann (cf. [10, (1.5)]), will play a crucial rôle in the proof of our main result.

Proposition 1.1 *Let $g : B \longrightarrow C$ be a surjective, properly irreducible homomorphism between two indecomposable modules such that $[C]$ belongs to a component of tree class A_∞^∞ or D_∞. If V is a Λ-module such that there exists a projective module P with $\tau(V) \oplus P \cong V$, then one of the following holds :*

(1) There is an embedding $ker\, g \hookrightarrow V$, or

(2) every map $\phi : ker\, g \longrightarrow \Omega^{-1}(V)$ which does not factor through a projective module is a monomorphism. $\square$

For future reference we also record the dual version of (1.1) :

Proposition 1.2 *Let $g : C \longrightarrow B$ be an injective, properly irreducible homomorphism between two indecomposable modules such that $[C]$ belongs to a component of tree class A_∞^∞ or D_∞. If V is a Λ-module such that $\tau^{-1}(V) \oplus P \cong V$ for some projective module P, then one of the following holds :*

(1) There is a surjection $V \longrightarrow coker\, g$, or

(2) every map $\phi : \Omega(V) \longrightarrow coker\, g$ which does not factor through a projective module is an epimorphism. $\square$

2. Support Varieties of Nonperiodic Components

Throughout the remainder of the paper $(L, [p])$ is assumed to be a finite dimensional restricted Lie algebra, defined over an algebraically closed field F of characteristic $p > 0$. We fix a linear form $\chi \in L^*$ and consider the stable Auslander-Reiten quiver $\Gamma_s(L, \chi)$ of the reduced enveloping algebra $u(L, \chi)$. By definition, the algebra $u(L, \chi)$ is obtained from the ordinary enveloping algebra $\mathcal{U}(L)$ by factoring out the ideal generated by

$$\{x^p - x^{[p]} - \chi(x)^p 1 \ ; \ x \in L\}.$$

In case of $\chi = 0$, $u(L) := u(L, 0)$ is the restricted enveloping algebra of L whose stable Auslander-Reiten quiver will be denoted $\Gamma_s(L)$. According to [36, (V.4.3)] and [17, (1.2)]