Graphs of some CAT(0) complexes

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Abstract. In this note, we characterize the graphs (1–skeletons) of some piecewise Euclidean simplicial and cubical complexes having nonpositive curvature in the sense of Gromov's CAT(0) inequality. Each such cell complex K is simply connected and obeys a certain flag condition. It turns out that if, in addition, all maximal cells are either regular Euclidean cubes or right Euclidean triangles glued in a special way, then the underlying graph $G(K)$ is either a median graph or a hereditary modular graph without two forbidden induced subgraphs. We also characterize the simplicial complexes arising from bridged graphs, a class of graphs whose metric enjoys one of the basic properties of CAT(0) spaces. Additionally, we show that the graphs of all these complexes and some more general classes of graphs have geodesic combings and bicombings verifying the 1– or 2–fellow traveller property.

1. Introduction

The idea of formulating nonpositive curvature for general geodesic metric spaces goes back to work of Alexandrov and Busemann [1, 2, 18]. Gromov [31] introduced the term CAT(0) for simply connected geodesic metric spaces in which the geodesic triangles are thinner than in the Euclidean plane. CAT(0) spaces have many nice equivalent definitions, reflecting different geometric features [6, 16, 17, 47]. A particularly important class of such spaces is formed by CAT(0) simplicial and cubical complexes [16]. Gromov [31] has shown that CAT(0) cubical complexes can be characterized in a purely combinatorial way; see [16, 17, 20, 21, 47] for related results and applications. Further properties of such complexes have been presented in the paper of Sageev [51] (he called them cubings). Niblo and Reeves [44] further investigate the properties of cubings and show how to find a bicombing of a Caley graph of their fundamental groups. On the other hand, Gersten and Short [29, 29] use the CAT(0) singular disk diagrams to provide an authomatic structure for fundamental groups of some two dimensional piecewise Euclidean complexes. Other examples of CAT(0) polygonal complexes were given in [16, 17, 31, 47].

In this article we show that certain CAT(0) simplicial or cubical cell complexes can be characterized by their underlying graphs. In particular, we show that the graphs of cubings are exactly the median graphs. They and the associated cubical complexes have many nice geometric, convexity and algebraic properties already known before [11, 35, 42, 43, 55]. This allows to simplify some results from [51]. Also we characterize the graphs of CAT(0) 2–complexes whose cells are right triangles as graphs in which all isometric cycles have length four and do not contain two forbidden subgraphs. Next we present a characterization of simplicial complexes arising from graphs in which all isometric cycles have length three (so–called bridged graphs). It is known that bridged graphs share one of the main properties of CAT(0) spaces, namely the convexity of the neighbourhoods of convex sets. We present examples of simplicial complexes with regular cells arising from bridged graphs which are not CAT(0). Nethertheless, in few important cases we obtain CAT(0) complexes. An important feature of bridged graphs is that they graph–metric admits a bicombing satisfying the k–fellow traveller condition for $k \leq 2$. This property (for different $k \leq 3$) is shared by all classes of graphs arising in this note, in particular, by folder graphs, median graphs (underlying graphs of cubings), and Helly graphs, a class of graphs containing the Caley graphs fundamental groups of cubings.

The paper is organized as follows. In Section 2 we will establish the notation and definitions used in the rest of the paper. In Section 3 we recall the basic characterizations of CAT(0) complexes; in Section 4 we introduce the classes of graphs and recall their main properties. In Section 5 we briefly disscuss the minimal disks diagrams which are used in the subsequent proofs. In Section 6 we establish the correspondence between the cubings and median complexes. Using this result we derive some properties of cubings. In Section 7 we focus our attention on CAT(0) 2–complexes whose cells are right triangles (we call them folder complexes). We characterize folder complexes and present their basic properties which can be derived from the specificity of their skeletons. In Section 8 we establish that the flag complexes whose graphs are bridged are exactly the simply connected simplicial complexes in which the links of vertices do not contain induced 4–cycles and 5–cycles. Finally, in Section 9 we present the results about combings and bicombings of the underlying graphs of complexes under consideration.

2. Preliminaries

This section summarizes some preliminaries from topology and metric spaces. Basic notions and results from topology can be found in any textbook on combinatorial topology (e.g. [52]).

2.1 Simplicial and cubical complexes

Let K be an abstract *simplicial complex,* i.e., a collection of sets (called *simplexes*) such that $\sigma \in K$ and $\sigma' \subseteq \sigma$ implies $\sigma' \in K$. A *cubical complex* is a set K of cubes of any dimensions which is closed under taking subcubes and nonempty intersections. We denote by $V(K)$ and $E(K)$ the *vertex set* and the *edge set* of K, namely, the set of all 0–simplexes and 1–simplexes of K. The pair $(V(K), E(K))$ is called the *(underlying) graph* or the 1–*skeleton* of K and is denoted by $G(K)$ or K^1 (the underlying graph of a cubical complex is defined similarly). More generally, for each $r \geq 0$ the r–*skeleton* of K is the simplicial complex $K^r = \{\sigma \in K : \#\sigma \leq r+1\}$. A *geometric realization* $|K|$ of K is a polyhedron obtained by the following construction. Let $\phi : V(K) \to \mathbb{R}^k$ be an injection such that for any two simplexes σ', σ'', $\mathrm{conv}\phi(\sigma') \cap \mathrm{conv}\phi(\sigma'') = \mathrm{conv}\phi(\sigma' \cap \sigma'')$. Then $|K| = \cup\{\mathrm{conv}\phi(\sigma) : \sigma \in K\}$.

The *link* of a simplex σ in K, denoted $\mathrm{link}(\sigma, K)$ is the abstract simplicial complex consisting of all simplexes σ' such that $\sigma \cap \sigma' = \emptyset$ and $\sigma \cup \sigma' \in K$. The *star* $\mathrm{star}(\sigma, K)$ of σ is the set of all simplexes containing σ. (The star of a cube in a cubical complex is defined analogously.) A simplicial complex K is a *flag complex* if any set of vertices is included in a member of K whenever each pair of its vertices is contained in a member of K. In the theory of hypergraphs this condition is called conformality. The link of any simplex in a flag complex is again a flag complex. A flag complex can be recovered by its underlying graph $G(K)$: the cliques of $G(K)$ are exactly the simplexes of K.