Connectedness of Opposite-Flag Geometries in Moufang Polygons

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Abstract

We show that the geometry of the elements opposite a certain flag in a Moufang polygon is always connected, up to some small cases. This completes the determination of all Moufang polygons for which this geometry is disconnected.

1 Introduction and statement of the Main Result

A *generalized n-gon*, $n \geq 2$, is a rank 2 geometry whose incidence graph has diameter n and girth $2n$, and each vertex has valency ≥ 3. If the latter condition is not satisfied , then we have a *weak generalized n-gon.* In this paper, we will always consider generalized n-gons with $n \geq 3$ (generalized 2-gons are trivial geometries). They are the irreducible spherical buildings of rank 2. A *generalized polygon* is a generalized n-gon, for some $n \geq 2$. We will view generalized polygons as geometries of rank 2 whose elements are points and lines. The *dual* is obtained by interchanging these names. A *flag* is an incident point-line pair and hence a chamber in the corresponding spherical rank 2 building. Generalized polygons were introduced by Tits [10] and are the basic rank 2 incidence geometries.

Let Γ be a generalized n-gon, $n \geq 3$. Given a fixed flag F in Γ, we define $\Gamma^{(F)}$ to be the set of all flags opposite F in Γ together with all points and lines occurring in these flags. So $\Gamma^{(F)}$ is a rank 2 subgeometry of Γ which we call

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opposite-flag geometry. The question arises what an opposite flag geometry looks like. In particular, is it connected? If Γ satisfies the Moufang condition, this plays an important role in the theory of twin buildings, see Condition (co) in MÜHLHERR & RONAN [6] and Condition $(*)$ in ABRAMENKO & MÜHLHERR [2]. In the present paper, we answer it for all such Moufang polygons. The result has already been used in the fundamental papers on twin buildings cited above, although there does not exist a complete proof in the literature. Let us briefly mention the history of the problem and its solution.

The question for arbitrary finite polygons was put to Andries Brouwer by the second author in connection with some characterizations of finite Moufang polygons. It was solved in BROUWER [3] by a matrix-technique. Around the same time, the first author determined the number of connected components of the opposite-flag geometries of all Moufang polygons (finite and infinite) using a group-theoretical argument (a sketch of this is contained in ABRAMENKO [1]). When the second author was confronted with this result, he managed to produce a geometric proof for the Moufang hexagons. In the present paper we present the original proof of the first author for Moufang octagons (relying on the classification of Moufang octagons, as given by TITS [15]), and we give two proofs for the case of Moufang hexagons: one entirely geometric and one group-theoretic, simplified by a geometric reduction. These proofs use the fact that infinite Moufang hexagons have at least 7, respectively 5 points on each line. This follows from the fact that appropriate Jordan algebras over finite fields are finite themselves. An alternative proof of the Main Result is given by Jacques Tits (lectures at Collège de France, 13 and 20 January 1998), based on Corollary 5 below. We comment on Tits' proof at the end of Subsection 3.1.

Our Main Result reads as follows.

Main Result. *Let Γ be a Moufang polygon, and let F be any flag of Γ. Then the geometry $\Gamma^{(F)}$ is connected, except in the following cases:*

(*i*) *Γ is the generalized quadrangle associated to the symplectic group $\mathbf{Sp}_4(2)$. In this case, $\Gamma^{(F)}$ has 2 connected components.*

(*ii*) *Γ is the generalized hexagon (or its dual) associated to the group $\mathbf{G}_2(2)$. In this case, $\Gamma^{(F)}$ has 4 connected components.*

(*iii*) *Γ is the generalized hexagon associated to the group $\mathbf{G}_2(3)$. In this case, $\Gamma^{(F)}$ has 3 connected components.*