Game chromatic number of graphs

Thomas Dinski
Department of Mathematics
Universität Bielefeld
dinski@mathematik.uni-bielefeld.de

Xuding Zhu*
Department of Applied Mathematics
National Sun Yat-sen University, Taiwan
zhu@ibm7.math.nsysu.edu.tw

†

Abstract

We show that if a graph has acyclic chromatic number k, then its game chromatic number is at most $k(k+1)$. By applying the known upper bounds for the acyclic chromatic numbers of various classes of graphs, we obtain upper bounds for the game chromatic number of these classes of graphs. In particular, since a planar graph has acyclic chromatic number at most 5, we conclude that the game chromatic number of a planar graph is at most 30, which improves the previous known upper bound for the game chromatic number of planar graphs.

*This research was supported in part by the National Science Council of the Republic of China under grant NSC87-2115-M-110-004.

†*1991 Mathematics Subject Classification.* 05C20, 05C35, 05C15
Key words and phrases. Acyclic chromatic number, game chromatic number, planar graph, partial k-tree
Submitted to *Discrete Mathematics*

1 Introduction

Let G be a finite graph and let X be a set of colors. We consider a modified graph coloring problem posed as a two-person game, with one person (Alice) trying to color a graph, and the other (Bob) trying to prevent this from happening. Alice and Bob alternate turns, with Alice having the first move. A move consists of selecting a previously uncolored vertex x and assigning to it a color from the color set X distinct from the colors assigned previously (by either player) to neighbours of x. If after $n = |V(G)|$ moves, the graph G is colored, Alice is the winner. Bob wins if an impass is reached before all nodes in the graph are colored, i.e., for every uncolored vertex x and every color α from X, x is adjacent to a vertex having color α. The *game chromatic number* of a graph $G = (V, E)$, denoted by $\chi_g(G)$, is the least cardinality of a color set X for which Alice has a winning strategy. This parameter is well-defined, since Alice always wins if $|X| = |V|$.

The acyclic chromatic number of a graph is another variation of the chromatic number of a graph. Suppose $G = (V, E)$ is a graph. The acyclic chromatic number of G, denoted by $\chi_a(G)$, is the least number t so that the vertices of G can be colored by t colors in such a way that each color class is an independent set, and the subgraph of G induced by any two color classes is acyclic, i.e., the union of every two color classes induces a forest.

The acyclic chromatic number of a graph was introduced by Grünbaum [3]. It was conjectured by Grünbaum [3], and proved by Borodin [2], that the maximum acyclic chromatic number of a planar graph is equal to 5.

The game chromatic number of a graph was first studied by Faigle, Kern, Kierstead and Trotter [4]. It was proved by Kierstead and Trotter [5] that the maximum of the game chromatic number of a forest is 4; the maximum game chromatic number a planar graph is between 8 and 33. It was also proved by Kierstead and Trotter [5] that there is a function $f(n)$ such that any graph embeddable on the surface of orientable genus n have game chromatic number at most $f(n)$. However, no explicit formulae for $f(n)$ is given.

This paper explores the relation between the game chromatic number and the acyclic chromatic number of a graph. We shall prove that if a graph has acyclic chromatic number k, then its game chromatic number is at most $k(k+1)$. This implies, in particular, that the game chromatic number of a planar graph is at most 30. For $n \geq 1$, it was proved by Albertson and Berman [1] that a graph embeddable in an orientable surface of genus n has acyclic chromatic number at most $4n+4$. Therefore the game chromatic number of a graph embeddable in an oriantable surface of genus n has game chromatic number at most $(4n+4)(4n+5)$. We also pose some open problems.