Finitely Generated Profinitely Dense Free Groups in Higher Rank Semi–simple Groups.

by

G.A. Soifer[1]
Math. and Computer Sciences Dept.
University of Bar–Ilan
Ramat–Gan, Israel
e-mail: soifer@sunlight.cs-ac.il

T.N. Venkataramana
School of Mathematics
Tata Institute of
Fundamental Research
Homi Bhabha Road, Colaba
Bombay 400 005, India
e-mail: venky@math.tifr.res.in

Introduction

We recall the main result from [Mar–Soi]. There it is proved that if Γ is a finitely generated linear group which is not an extension of a solvable group by a finite group, then Γ contains (uncountably) many maximal subgroups of infinite index. The strategy there is to produce free (*infinitely* generated) subgroups F_∞ of Γ which are "profinitely dense" in Γ (i.e. map *onto* every finite quotient of Γ). Any maximal subgroup of Γ which contains such an F_∞ must then necessarily be of infinite index.

Aner Shalev has asked the following question: can one obtain *finitely* generated free subgroups F of Γ which are profinitely dense in Γ (we will refer to such groups as $f \cdot g \cdot p \cdot d \cdot f$ subgroups).

A result of [Scott] says that if Γ is a "Surface group" (i.e. the fundamental group of a compact Riemann surface of genus $\geqq 2$) then *any* finitely generated subgroup of Γ is contained in a *proper* subgroup of finite order in Γ . Thus the above question has a negative answer for general subgroups of $GL_n(\mathbb{C})$.

[1]Partially supported by GIF grant No. G–454–213.06/95

In this paper we prove (see *Theorem* (3.1)) that if Γ is an arithmetic subgroup of a non–compact linear semisimple group G such that the associated simply connected algebraic group over $\mathbb{Q}$ has the congruence subgroup property (see (1.2) for the relevant definition), then Γ contains a finitely generated profinitely dense free subgroup. As a corollary (see (3.3)) we obtain a $f \cdot g \cdot p \cdot d \cdot f$ subgroup of $SL_n(\mathbb{Z})$ $(n \geqq 3)$. More generally, we prove that if Γ is an irreducible arithmetic non–cocompact lattice in a higher rank group (see (3.2)) then Γ contains $f \cdot g \cdot p \cdot d \cdot f$ groups.

A Theorem of [Nori] and [Weisfeiler], together with a Theorem of [Tits 1] on the existence of free groups in linear groups already implies the existence of $f \cdot g \cdot p \cdot d \cdot f$ subgroups of *some subgroup* of finite index in our group Γ . We use the methods of [Mar–Soi] to construct a free group of this type and *extend* this free group to the full group Γ . More precisely, we construct, following [Mar–Soi], certain "free–systems" which can be extended to larger free systems (see (2.3)); these can be exploited to prove Theorem (3.1).

According to the general strategy of [Mar–Soi] these finitely generated profinitely dense free subgroups F of Γ can be embedded in maximal subgroups of infinite index in Γ . It would be of interest to know if these maximal subgroups are finitely generated. In contrast, maximal subgroups of infinite index in lattices of $SL_2(\mathbb{R})$ are infinitely generated ([Soifer]).

We are very grateful to Professor G.A. Margulis for helpful conversations and to Professor H. Abels and the department of Mathematics of Bielefeld University for their warm hospitality. We also thank Professor L. Potyagailo for useful references and comments.

§ 1. Preliminary Reductions and A Theorem of [Nori] [Weisfeiler].

(1.1) Notation: Let G be a real non–compact semi–simple linear Lie group and $\Gamma \subset G$ an *arithmetic* lattice. By definition this means the following. There exists a connected simply connected semi–simple algebraic group $\mathcal{G}$ defined over