Reciprocity laws for simple algebras over function fields of number curves

by

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1 Introduction

Let k be a field of characteristic zero, X a smooth projective curve over k, X^1 the set of all closed points of X and $\overline{X} = X \times_k \overline{k}$ for an algebraic closure $\overline{k}$ of k. The following sequence of abelian groups and homomorphisms is well known

$$0 \to \mathrm{Br}\, X \to \mathrm{Br}\, k(X) \stackrel{\mathrm{inv}}{\to} \bigoplus_{x \in X^1} H^1(k(x), \mathbf{Q}/\mathbf{Z}) \stackrel{\Sigma}{\to} H^1(k, \mathbf{Q}/\mathbf{Z})$$

where $\mathrm{Br}\, X$ is the Brauer group of X, $\mathrm{Br}\, k(X)$ is the Brauer group of the function field $k(X)$ on X, $k(x)$ is the residue field at $x \in X^1$ and $H^i(k, ?)$ denotes the Galois cohomology of k. Later on we shall present a cohomological interpretation of this sequence, but now we only note that it allows us to study $\mathrm{Br}\, k(X)$ using the local-to-global principle. The homomorphism inv is induced by the local ones $\mathrm{inv}_x : \mathrm{Br}\, k(X) \to H^1(k(x), \mathbf{Q}/\mathbf{Z})$. If $a \in \mathrm{Br}\, k(X)$, then the character $\mathrm{inv}_x(a)$ is called the local invariant of a at x and its order equals the ramification index at x of the algebras in a. Hence ker(inv) coincides with the unramified Brauer group $\mathrm{Br}_{nr}\, k(X)$ of $k(X)$ which is canonically isomorphic to $\mathrm{Br}\, X$ [1]. The homomorphism Σ sums corestrictions of characters. We shall say that an element from $\bigoplus_{x \in X^1} H^1(k(x), \mathbf{Q}/\mathbf{Z})$ satisfies a reciprocity law if it belongs to $\ker(\Sigma)$. It can be shown (see below), that the complete system of local invariants of any $a \in \mathrm{Br}\, k(X)$ satisfies the reciprocity law, i.e. $\mathrm{im(inv)} \subset \ker(\Sigma)$. In some cases the converse statement is also true. For example, this holds if $X \cong \mathbf{P}^1$. In other words, in this case the group of local invariants of central simple algebras over $k(X)$ is comletely determined by the reciprocity law. But in general $\ker(\Sigma) \neq \mathrm{im(inv)}$ (see [2], [3]). This means that there exists an "additional reciprocity law" describing local invariants of central simple algebras over $k(X)$. Therefore we are lead to study the quotient

$$\Delta := \ker(\Sigma)/\,\mathrm{im(inv)}\ .$$

We shall call it the obstruction group for realizing characters satisfying the reciprocity law as systems of local invariants of algebras over $k(X)$, or shortly the obstruction group.

In 1975 B.Becker has proved ([4]) that for a curve X defined over a number field k with at least one k-rational point Δ is a group of exponent 2 which is trivial if k is totally imaginary. The case where X is a genus one curve was considered in [2] and [3]. It has been proved there that $\Delta \cong (\mathbf{Z}/2)^s$ where s is the number of real embeddings of k which leads to curves with two-component Jacobians, and that $\ker(\Sigma)$ is the direct sum of $\mathrm{im}(\mathrm{inv})$ and Δ.

In this paper we generalize the results above to an arbitrary smooth projective curve X of positive genus defined a over number field k. More precisely, we prove the following

Main Theorem. *Let k be a number field, $s = 0$ if k is totally imaginary, and s the number of real embeddings $\rho_i : k \hookrightarrow \mathbf{R}$, $i = 1, \dots, s$, for k otherwise. Let X be a smooth projective curve of arbitrary genus defined over k with a k-point, and $X_i = X \times_k \mathbf{R}$ where the product is taken with respect to ρ_i. Let also r_i be the number of components of $X_i(\mathbf{R})$ in the real topology. Then*

$$\Delta \cong (\mathbf{Z}/2)^{r_1+\ldots+r_s-s} .$$

Moreover, the short exact sequence $0 \to \mathrm{im}(\mathrm{inv}) \to \ker(\Sigma) \to \Delta \to 0$ splits, so

$$\ker(\Sigma) \cong \mathrm{im}(\mathrm{inv}) \oplus (\mathbf{Z}/2)^{r_1+\ldots+r_s-s} .$$

Notation and conventions. For any abelian group A and any natural n we denote by $A \xrightarrow{n} A$ the multiplication by n, ${}_nA$, nA and A/n are its kernel, image and cokernel respectively. $A^n = A \oplus \ldots \oplus A$ is the direct sum of n copies of A and let $A^0 = 0$. A variety over k is a geometrically integral separated scheme of finite type over k. If X is a variety over k and L/k is a field extension, then $X_L := X \times_k L$, $\overline{X} = X \times_k \overline{k}$ and $X(L)$ is the set of closed L-points of X_L. We will denote by G the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ of k.

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